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Solvable groups definable in o-minimal structures

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Abstract

In this paper we develop group extension theory over an o-minimal structure $\mathcal{N}$ and use it to describe $\mathcal{N}$-definable solvable groups. We prove an o-minimal analogue of the Lie–Kolchin–Mal'cev theorem and we describe $\mathcal{N}$-definable G-modules and $\mathcal{N}$-definable rings.

MSC: 03C64; 20E99

1. Introduction

We work inside an o-minimal structure $\mathcal{N} = (N, <, \ldots)$ and therefore definable means $\mathcal{N}$-definable. We assume the reader's familiarity with basic o-minimality (see [3]). We start by recalling some basic notions and results on definable groups that will be used through the paper.

Hrushovski gives in [9] a proof of Weil's Theorem that an algebraic group can be recovered from birational data. This proof is adapted by Pillay in [24] (see Proposition 2.5 in [24]) to show that a definable group G can be equipped with a unique definable manifold structure making the group into a topological group, and that definable homomorphisms between definable groups are topological homomorphisms. In fact, as remarked in [19, Fact 1.10 and Lemma 1.11], if $\mathcal{N}$ is an o-minimal expansion of a real closed field then G, equipped with the above unique definable manifold structure, is a C^p group for all $p \in \mathbb{N}$; and definable homomorphisms between definable groups are C^p homomorphisms for all $p \in \mathbb{N}$. Moreover, by [19, Lemma 2.17], the definable manifold structure on a definable subgroup is the sub-manifold structure.

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By [24] Remark 2.13(ii) definable groups satisfy the descending chain condition (DCC) on definable subgroups. This is used to show that the definably-connected component of the identity G^0 of a definable group G is the smallest definable subgroup of G of finite index ([24, Proposition 2.12]). Also an infinite definable group G has an infinite definable abelian subgroup ([24, Corollary 2.15 (i)]). Any definable subgroup H of G is closed and the following are equivalent (see [24, Corollary 2.8 and Lemma 2.11]):

(i) H has finite index in G.
(ii) $\dim H = \dim G$.
(iii) H contains an open neighbourhood of the identity element of G.
(iv) H is open in G.

Finally, by [27, Lemma 5.7 and Corollary 5.8] an infinite abelian definable group G has unbounded exponent and the subgroup $\mathrm{Tor}(G)$ of torsion points of G is countable. In particular, if $\mathcal{N}$ is $\aleph_0$-saturated then G has an element of infinite order.

One-dimensional definable manifolds are classified in [25, Proposition 2] and the following is deduced. Suppose that G is a one-dimensional definably-connected definable group. Then by [24, Corollary 2.15 (ii)] G is abelian, and G is either torsion-free or for each prime p the set of p-torsion points of G has p elements. In the former case G is an ordered abelian divisible definably simple definable group.

Note that if I is a one-dimensional definably-connected ordered definable group, then the structure $\mathcal{I}$ induced by $\mathcal{N}$ on I (that is, for all $n \geqslant 1$ the $\mathcal{I}$-definable subsets of I^n are the definable subsets of I^n) is an o-minimal structure with domain I. In particular, we have the following results from [14] (see Theorem A, Proposition 3.3 and Theorem C respectively). Suppose that $(I, 0, 1, +, <)$ is a one-dimensional definably-connected, torsion-free, definable group, where 1 is a fixed positive element. Let $\Lambda(\mathcal{I})$ be the division ring of all $\mathcal{I}$-definable endomorphisms of $(I, 0, +)$. Then exactly one of the following holds:

(1) $\mathcal{I}$ is *linearly bounded* with respect to $+$ (i.e, for every $\mathcal{I}$-definable function $f : I \to I$ there is $r \in \Lambda(\mathcal{I})$ such that $\lim_{x\to+\infty} [f(x) - rx] \in I$).

(2) There is an $\mathcal{I}$-definable binary operation $\cdot$ such that $(I, 0, 1, +, \cdot, <)$ is a real closed field.

Also, up to $\mathcal{I}$-definable isomorphism, there is at most one $\mathcal{I}$-definable group $(I, 0, *)$ such that $\mathcal{I}$ is linearly bounded with respect to $*$ and at most one $\mathcal{I}$-definable (real closed) field $(I, 0, 1, \oplus, \otimes)$. Moreover, by [14, Proposition 3.2 and Lemma 1.7], the following are equivalent:

(i) $\mathcal{I}$ is linearly bounded with respect to $+$.
(ii) For every $\mathcal{I}$-definable function $f : A \times I \to I$, where $A \subseteq I^n$, there are $r_1, \dots, r_l \in \Lambda(\mathcal{I})$ such that for every $a \in A$ there is $i \in \{1, \dots, l\}$ with $\lim_{x\to+\infty} [f(a,x) - r_i x] \in I$.
(iii) There is no infinite definable subset of $\Lambda(\mathcal{I})$.

Let $(I, 0, 1, +, <)$ be as above and let $\Lambda = \Lambda(\mathcal{I})$. Then $\mathcal{I}$ is called *semi-bounded* if every $\mathcal{I}$-definable set is already definable in the reduct

$$(I, 0, 1, +, <, (B_k)_{k\in K}, (\lambda)_{\lambda\in\Lambda}),$$

of $\mathscr{I}$ where $(B_k)_{k\in K}$ is the collection of all bounded $\mathscr{I}$-definable sets. According to [4, Fact 1.6], the following are equivalent:

(i) $\mathscr{I}$ is semi-bounded.
(ii) There is no $\mathscr{I}$-definable function between a bounded and an unbounded subinterval of I.
(iii) There is no $\mathscr{I}$-definable (real closed) field with domain an unbounded subinterval of I (equivalently there is no $\mathscr{I}$-definable (real closed) field with domain I).
(iv) For every $\mathscr{I}$-definable function $f : I \to I$ there are $r \in \Lambda$, $x_0 \in I$ and $c \in I$ such that $f(x) = rx + c$ for all $x > x_0$.
(v) $\mathscr{I}$ satisfies the "structure theorem".

Note that by the remarks above, if $\mathscr{I}$ is semi-bounded, then up to $\mathscr{I}$-definable isomorphism, $(I, 0, +)$ is the only $\mathscr{I}$-definable group with domain I. In this case we call $(I, 0, +)$ the *additive group of* $\mathscr{I}$.

Let $(I, 0, 1, +, \cdot, <)$ be a real closed field definable in $\mathscr{N}$. Let $\mathscr{K}(\mathscr{I})$ be the ordered field of all $\mathscr{I}$-definable endomorphisms of the multiplicative group $(I^{>0}, \cdot, 1)$. The field addition on $\mathscr{K}(\mathscr{I})$ is pointwise multiplication and the multiplication is composition. Note that the map from $\mathscr{K}(\mathscr{I})$ to I which sends α into $\alpha'(1)$ is an embedding of ordered fields. The elements of $\mathscr{K}(\mathscr{I})$ are called *power functions* and for $\alpha \in \mathscr{K}(\mathscr{I})$ with $\alpha'(1) = r$ we write $\alpha(x) = x^r$. By [13, Theorem 3.5] exactly one of the following holds:

(1) $\mathscr{I}$ is *power bounded* (i.e., for every $\mathscr{I}$-definable function $f : I \to I$ there is $r \in \mathscr{K}(\mathscr{I})$ such that ultimately $|f(x)| < x^r$).
(2) $\mathscr{I}$ is *exponential* (i.e., there is an $\mathscr{I}$-definable ordered group isomorphism $e : (I, 0, +, <) \to (I^{>0}, 1, \cdot, <)$).

Moreover, by [13, Theorem 4.1], the following are equivalent:

(i) $\mathscr{I}$ is power bounded.
(ii) For every $\mathscr{I}$-definable function $f : A \times I \to I$, where $A \subseteq I^n$, there are $r_1, \ldots, r_l \in \mathscr{K}(\mathscr{I})$ such that for every $a \in A$, if the function $x \to f(a,x)$ is ultimately nonzero then, there is $i \in \{1, \ldots, l\}$ with $\lim_{x\to+\infty} [f(a,x)/x^{r_i}] \in I$.
(iii) There is no infinite definable subset of $\mathscr{K}(\mathscr{I})$.

Note the following: if G is an $\mathscr{I}$-definably-connected, $\mathscr{I}$-definable, one-dimensional torsion-free group, then G is $\mathscr{I}$-definably isomorphic to an $\mathscr{I}$-definable group $(I, 0, *)$ with domain I. Also, as we saw before, there are (up to $\mathscr{I}$-definable isomorphism) at most two $\mathscr{I}$-definably-connected, $\mathscr{I}$-definable one-dimensional torsion-free groups (one of these groups is the additive group $(I, 0, +)$ of $\mathscr{I}$ and the other one, if it exists, is the unique $\mathscr{I}$-definable group $(I, 0, *)$ with respect to which $\mathscr{I}$ is linearly bounded). The *Miller–Starchenko conjecture* says that in an o-minimal expansion $\mathscr{I}$ of a field $(I, 0, 1, +, \cdot, <)$, every $\mathscr{I}$-definably-connected, $\mathscr{I}$-definable one-dimensional torsion-free group is $\mathscr{I}$-definably isomorphic to either $(I, 0, +)$ or $(I^{>0}, 1, \cdot)$. If $\mathscr{I}$ is power bounded, then since $(I, 0, +)$ and $(I^{>0}, 1, \cdot)$ are not $\mathscr{I}$-definably isomorphic, the conjecture holds.

Suppose that the Miller–Starchenko conjecture does not hold for $\mathscr{I}$. Then $\mathscr{I}$ is exponential, and we call the unique $\mathscr{I}$-definable group $G=(I,0,*)$ which is not $\mathscr{I}$-definably isomorphic to $(I,0,+)$ or $(I^{>0},1,\cdot)$ the *Miller–Starchenko group of* $\mathscr{I}$. Note the following (see Lemma 5.4): $\alpha : G \to (I,0,+)$ is an abstract C^1 isomorphism iff for all $s \in G$, we have $\alpha'(s)\frac{\partial *}{\partial x}(0,s) = \alpha'(0)$ where, for all $t,s \in G$, we set $*(t,s) = t * s$. This says exactly that α is Pfaffian over $(I,0,1,+,\cdot,*,<)$ in the sense of [26]. (Note that, by associativity of $*$, for all $s \in G$, we have $\frac{\partial *}{\partial x}(0,s) \neq 0$).

We now describe the main results of this paper, starting with a preliminary definition.

Definition 1.1. Let $\mathscr{I} = (I, <_I, \ldots)$ and $\mathscr{J} = (J, <_J, \ldots)$ be two o-minimal structures definable in $\mathscr{N}$. We say that $\mathscr{I}$ and $\mathscr{J}$ are *globally orthogonal* if there is no definable bijection between I and J.

The trichotomy theorem from [21] and the theory of non-orthogonality from [19] are used to prove the following:

Corollary 3.11. *Let U be a definable group and A a definable normal subgroup of U. Then there is a definable extension $1 \to A \to U \xrightarrow{j} G \to 1$ with a definable section $s : G \to U$.*

If in Corollary 3.11 we take A to be the definable radical of U, i.e. the maximal definably-connected, definable solvable normal subgroup of U, we get that G is either finite or definably semi-simple, i.e. it has no infinite proper abelian definable normal subgroup. Definably semi-simple definable groups are classified in [19, Theorem 4.1] (see also [19,21]). We denote by $\mathbb{G}$ the structure $(G,\cdot)$ where $\cdot$ is the group operation of G.

Theorem 1.2 (Peterzil et al. [19]). *Suppose that G is a $\mathbb{G}$-definably-connected, definably semi-simple definable group. Then $G = G_1 \times \cdots \times G_l$ and for each $i \in \{1,\ldots,l\}$ there is an o-minimal expansion $\mathscr{I}_i$ of a real closed field definable in $\mathscr{N}$ such that, for all $j \neq i$, $\mathscr{I}_j$ is globally orthogonal to $\mathscr{I}_i$ and G_i is $\mathscr{I}_i$-definably isomorphic to a I_i-semi-algebraic subgroup of $GL(n_i,I_i)$ which is a direct product of I_i-semi-algebraically simple, I_i-semi-algebraic subgroups of $GL(n_i,I_i)$.*

Theorem 1.2 together with Corollary 3.11, reduces the classification of definable groups to the classification of definable solvable groups. Corollary 3.11 allows us to develop group extension theory with abelian and non-abelian kernel over $\mathscr{N}$. We use this theory to prove the results below for definable solvable groups. But before outlining these we need two more definitions.

Definition 1.3 (Peterzil and Steinhorn [23]). Let G be a definable group. We say that G is *definably compact* if for every definable continuous embedding $\sigma:(a,b) \subseteq N \to G$, where $-\infty \leqslant a < b \leqslant +\infty$, there are $c,d \in G$ such that $\lim_{x\to a^+} \sigma(x) = c$ and $\lim_{x\to b^-} \sigma(x) = d$, where the limits are taken with respect to the topology on G.

Definition 1.4. Let $\mathscr{I}=(I,<_I,\ldots)$ be an o-minimal structure definable in $\mathscr{N}$. We say that an $\mathscr{I}$-definable abelian group U *has no $\mathscr{I}$-definably compact parts* if there are $\mathscr{I}$-definable subgroups $1=U_0<U_1<\cdots<U_n=U$ such that, for each $j\in\{1,\ldots,n\}$, the group U_j/U_{j-1} is a one-dimensional $\mathscr{I}$-definably-connected, torsion-free $\mathscr{I}$-definable group. We say that an $\mathscr{I}$-definable solvable group U *has no $\mathscr{I}$-definably compact parts* if U has $\mathscr{I}$-definable subgroups $1=U_0 \trianglelefteq U_1 \trianglelefteq \cdots \trianglelefteq U_n=U$ such that, for each $i\in\{1,\ldots,n\}$, the group U_i/U_{i-1} is an $\mathscr{I}$-definable abelian group with no $\mathscr{I}$-definably compact parts.

Finally, we say that a definable solvable group U *has no definably compact parts* if U has no $\mathscr{N}$-definably compact parts.

Here we prove the following result about definably compact definable groups. This result already appeared in [22, Corollary 5.4] but under the additional assumption that $\mathscr{N}$ has definable Skolem functions (with a proof using the theory of $\bigvee$-definable groups). Here we give a more direct proof with no assumptions on $\mathscr{N}$.

Corollary 4.8. *Let U be a definably compact, definably-connected definable group. Then U is either abelian or $U/Z(U)$ is a definably semi-simple definable group. In particular, if U is solvable then it is abelian.*

Corollary 4.8 and the next result reduce the classification of definable solvable groups to the classification of definably compact definable abelian groups and of $\mathscr{I}$-definable solvable groups with no $\mathscr{I}$-definably compact parts. Here $\mathscr{I}$ is an o-minimal expansion of a real closed field definable in $\mathscr{N}$.

Theorem 5.8. *Suppose that U is a definably-connected definable solvable group. Then U has a definable normal subgroup V such that U/V is a definably compact definable solvable group and $V=K\times W_1\times\cdots\times W_s\times V_1\times\cdots\times V_k$. Here K is the definably-connected definably compact normal subgroup of U of maximal dimension. For each $j\in\{1,\ldots,s\}$ (resp., $i\in\{1,\ldots,k\}$), there is a semi-bounded o-minimal expansion $\mathscr{J}_j$ of a group (resp., an o-minimal expansion $\mathscr{I}_i$ of a real closed field) definable in $\mathscr{N}$ all of which are pairwise globally orthogonal such that W_j is a direct product of copies of the additive group of $\mathscr{J}_j$ and V_i is definably isomorphic to an $\mathscr{I}_i$-definable solvable group with no $\mathscr{I}_i$-definably compact parts.*

The next result describes $\mathscr{I}$-definable solvable groups with no $\mathscr{I}$-definably compact parts where $\mathscr{I}=(I,0,1,+,\ldots,<,\ldots)$ is an arbitrary o-minimal expansion of a real closed field.

Theorem 5.10. *Let $\mathscr{I}=(I,0,1,+,\cdot,<,\ldots)$ be an o-minimal expansion of a real closed field and let U be an $\mathscr{I}$-definable solvable group with no $\mathscr{I}$-definably compact parts. Then $U=W\times V$ where W is the maximal $\mathscr{I}$-definable subgroup of U which is a direct product of copies of the linearly bounded one-dimensional torsion-free $\mathscr{I}$-definable group. The group V is an $\mathscr{I}$-definable group such that $Z(V)$ has an $\mathscr{I}$-definable subgroup Z such that $Z(V)/Z$ is a direct product of copies of the linearly* *bounded one-dimensional torsion-free $\mathcal{I}$-definable group. There are $\mathcal{I}$-definable subgroups $1 < Z_1 < \cdots < Z_m = Z$ such that, for each $l \in \{1,\dots,m\}$, the group Z_l/Z_{l-1} is the additive group of $\mathcal{I}$, and there is an $\mathcal{I}$-definable embedding of $V/Z(V)$ into $GL(n,I)$.*

Peterzil and Steinhorn ask in [23] if a definable abelian group U of dimension two and with no definably compact parts is a direct product of one-dimensional definably-connected torsion-free definable groups. For solvable definable groups with no definably compact parts, Theorems 5.8 and 5.10 above reduce this problem to the case where U is an $\mathcal{I}$-definable group, $\mathcal{I}$ is an o-minimal expansion of a real closed field $(I,0,1,+,\cdot,<)$ definable in $\mathcal{N}$ and we have an $\mathcal{I}$-definable extension $1 \to A \to U \to G \to 1$ where $A=(I,0,+)$ and $G=(I,0,*)$ is a one-dimensional torsion-free $\mathcal{I}$-definable group. We prove (see Lemma 5.5) that in this case there is an $\mathcal{I}$-definable 2-cocycle $c \in Z^2_{\mathcal{I}}(G,A)$ for U such that U is $\mathcal{I}$-definably isomorphic to $A\times G$ iff there is an $\mathcal{I}$-definable function $\alpha: G \to A$ such that $\alpha'(s)\frac{\partial *}{\partial x}(0,s) = \alpha'(0) - \frac{\partial c}{\partial x}(0,s)$ for all $s \in G$.

Let $\mathcal{I}$ be an o-minimal expansion of a real closed field $(I,0,1,+,\cdot,<)$ and suppose that we have an abelian $\mathcal{I}$-definable extension $1 \to A \to U \to G \to 1$ (i.e., U is abelian) where $A=(I,0,+)$ and $G=(I,0,*)$ is a one-dimensional torsion-free $\mathcal{I}$-definable group. We shall say that U is a *Peterzil–Steinhorn $\mathcal{I}$-definable group* if U is not $\mathcal{I}$-definably isomorphic to $A\times G$.

A corollary of our main result is the following

Corollary 5.11. *Let $\mathcal{I}=(I,0,1,+,\cdot,<,\dots)$ be an o-minimal expansion of a real closed field with no Peterzil–Steinhorn $\mathcal{I}$-definable groups. Then every $\mathcal{I}$-definable solvable group U with no $\mathcal{I}$-definable compact parts is $\mathcal{I}$-definably isomorphic to a definable group of the form $U' \times G_1 \cdots G_k \cdot G_{k+1} \cdots G_l$ where U' is a direct product of copies of linearly bounded one-dimensional torsion-free $\mathcal{I}$-definable groups. For $i=1,\dots,k$, we have $G_i=(I,0,+)$ and for $i=k+1,\dots,l$, we have $G_i=(I^{>0},1,\cdot)$.*

From Corollary 5.11 we get the following result.

Corollary 5.12. *Let $\mathcal{I}$ and U be as in Corollary* 5.11. *Then there is an $\mathcal{I}$-definable embedding from $G=G_1\cdots G_k\cdot G_{k+1}\cdots G_l$ into some $GL(n,I)$. The group U is $\mathcal{I}$-definably isomorphic to a definable group in one of the reducts $(I,0,1,+,\cdot,\oplus)$, $(I,0,1,+,\cdot,\oplus,e^t)$ or $(I,0,1,+,\cdot,\oplus,t^{b_1},\dots,t^{b_r})$ of $\mathcal{I}$ where $(I,0,\oplus)$ is the Miller–Starchenko group of $\mathcal{I}$, e^t is the $\mathcal{I}$-definable exponential map* (*if it exists*), *and the t^{b_j}*'s *are $\mathcal{I}$-definable power functions. Moreover, if U is nilpotent then U is $\mathcal{I}$-definably isomorphic to a group definable in the reduct $(I,0,1,+,\cdot,\oplus)$ of $\mathcal{I}$.*

An application of Theorem 5.8 is the following result.

Theorem 7.2. *Let U be a definable group and let $\{T(x): x\in X\}$ be a definable family of non-empty definable subsets of U. Then there is a definable function $t: X \to U$ such that for all $x,y \in X$ we have $t(x)\in T(x)$ and if $T(x)=T(y)$ then $t(x)=t(y)$.*

This result shows that the many of the theorems from [22] can be obtained without the assumption that $\mathcal{N}$ has definable Skolem functions. We include here direct proofs (avoiding the use of $\bigvee$-definability theory) of some of these results, namely Corollary 4.8 above, Corollary 6.3 and Corollary 7.3.

In Section 6 we classify definable G-modules and use this to prove the o-minimal version of the Lie–Kolchin–Mal'cev theorem (see Theorem 6.9). In Section 8 we classify definable rings.

2. Definable quotients

Definition 2.1. Let S be a definable set and let $T=\{T(x):x\in X\}$ be a definable family of non-empty definable subsets of S. We say that T has *definable choice* if there is a definable function $t:X\to S$ such that $t(x)\in T(x)$ for all $x\in X$. If, in addition, t is such that for all $x,y\in X$, if $T(x)=T(y)$ then $t(x)=t(y)$, then we say that T has *strong definable choice*. The function t is called a (*strong*) *definable choice for the family T*. We say that the definable set S has (*strong*) *definable choice* if every definable family T of non-empty definable subsets of S has a (strong) definable choice.

The following are easy to prove.

Fact 2.2. (i) *If $f:R\to S$ is a definable map such that for all $s\in S$, $f^{-1}(s)$ is finite and S has* (*strong*) *definable choice then R has* (*strong*) *definable choice.* (ii) *If $g:S\to R$ is a surjective definable map and S has* (*strong*) *definable choice then R has* (*strong*) *definable choice.* (iii) *If $S=S_1\times\cdots\times S_k$ is definable and each S_i is definable and has* (*strong*) *definable choice then S has* (*strong*) *definable choice.*

For the proof of the next lemma we need to recall some definitions from [19]. An open interval $I\subseteq N$ is *transitive* if, for all $x,y\in I$, there are definably homeomorphic subintervals I_x,I_y of I containing x and y respectively. An open rectangular box $I_1\times\cdots\times I_n$ is transitive if all the intervals I_k are transitive.

Lemma 2.3. *A definable group U has a definable neighbourhood O of* 1 (*the identity*) *with strong definable choice.*

Proof. Since it is sufficient to prove the lemma for an ω_1-saturated elementary extension of $\mathcal{N}$, we will assume that $\mathcal{N}$ is ω_1-saturated.

By [19, Lemma 1.28], there is a definable chart (O',ϕ) on U at 1 such that $\phi(O')$ is a transitive rectangular box, say $I_1\times\cdots\times I_n$. Let $\phi(1)=(a_1,\ldots,a_n)$. Then by [21, Theorem 1.1], the definable structure $\mathcal{J}_i$ induced by $\mathcal{N}$ on some open subinterval J_i of I_i containing a_i is either an o-minimal expansion of a real closed field or an o-minimal expansion of an ordered divisible abelian partial group. Without loss of generality we may assume that $(J_i,a_i,+_i,-_i,<_i)$ is a definable ordered divisible abelian partial group with zero a_i and $J_i=(-_ie_i,e_i)$. Therefore, if $x\in J_i$, then there is a unique $y\in J_i$ denoted by $\frac{x}{2}$ such that $y+_i y=x$.

Let $J_i' = (-_i \frac{e_i}{2}, \frac{e_i}{2})$ and consider the definable functions l_i, r_i and m_i given by

$$l_i : J_i' \to J_i', \quad l_i(x) = x -_i \left| \frac{e_i -_i x}{2} \right|_i,$$

$$r_i : J_i' \to J_i', \quad r_i(x) = x +_i \left| \frac{e_i -_i x}{2} \right|_i,$$

$$m_i : J_i' \times J_i' \to J_i', \quad m_i(x) = x +_i \left| \frac{y -_i x}{2} \right|_i$$

where $| \ |_i$ is the natural norm in J_i. Since for all $x, y \in J_i'$, we have $l_i(x) <_i x$, $x <_i r_i(x)$ and if $x <_i y$ then $x <_i m_i(x, y) <_i y$, the definable set J_i' has strong definable choice.

By Fact 2.2(iii), $J_1' \times \cdots \times J_n'$ has strong definable choice and so, by Fact 2.2(ii), $O = \phi^{-1}(J_1' \times \cdots \times J_n')$ has strong definable choice. $\square$

Remark 2.4. The same argument shows that if $\mathscr{I} = (I, 0, +, <, \cdots)$ is a definable o-minimal expansion of an ordered group then for every $n \in \mathbb{N}$, the definable set I^n has strong definable choice and hence, by Fact 2.2(i), so does every definable subset of I^n.

Given a definable set S and a definable equivalence relation E on S, we will say that S/E is definable if there is a definable map $l : S \to T \subseteq S$ such that for all $x, y \in S$, $xEl(x)$ and xEy iff $l(x) = l(y)$. In this case the definable family $\{x/E : x \in S\}$ has a strong definable choice. If S is a definable group, E a definable normal subgroup and the set S/E is definable then S/E becomes in a natural way a definable group.

Theorem 2.5. *Let U be a definable group and let V be a definable normal subgroup of U. Then the definable family $\{xV : x \in U\}$ has a strong definable choice and so U/V is definable.*

Proof. Suppose that $U \subseteq N^m$ and for each $q \in \{0, \ldots, m\}$ let $\pi_q : N^m \to N^q$ be the projection onto the first q coordinates and let $\pi^q : N^m \to N$ be the projection onto the q-th coordinate.

Claim: For each $k \in \{0, \ldots, m\}$ there is a definable subset U_k of U such that (i) $\dim(U \setminus U_k) < \dim U$ and (ii) if $x \in U_k$ and $y \in U$ is such that $xV = yV$ then $y \in U_k$. Moreover, there are definable functions $l_1, \ldots, l_k : U_k \to N$ such that for each $x \in U_k$ there is $z \in xV$ such that $\pi_k(z) = (l_1(x), \ldots, l_k(x))$ and for all $y \in U$ if $xV = yV$ then $(l_1(x), \ldots, l_k(x)) = (l_1(y), \ldots, l_k(y))$.

The existence of a strong definable choice $l = (l_1, \ldots, l_m)$ for the family $\{xV : x \in U\}$ follows from this claim. The claim immediately implies the existence of l on a large definable subset U_m of U (i.e., $\dim(U \setminus U_m) < \dim U$). But by [24, Lemma 2.4], there are $u_1, \ldots, u_n \in U$ such that $U = u_1 U_m \cup \cdots \cup u_n U_m$ and so we can extend l from U_m to U inductively as follows. If $x \in u_1 U_m$, then we put $l(x) = u_1 l(u_1^{-1} x)$. Having extended l to $u_1 U_m \cup \cdots \cup u_k U_m$, define l on $u_{k+1} U_m$ by $l(x) = u_{k+1} l(u_{k+1}^{-1} x)$ if $xV \cap (u_1 U_m \cup \cdots \cup u_k U_m) = \emptyset$ and $l(x) = l(y)$ for some (for all) $y \in xV \cap (u_1 U_m \cup \cdots \cup u_k U_m)$ otherwise.

Proof of Claim. We use induction on k. For $k=0$, set $U_0=U$ and let $l_0: U_0 \to N^0$ be the unique map.

Suppose that the claim is true for k. We will show that it is true for $k+1$. For this consider the definable family $\{V_k(x): x \in U_k\}$ of non-empty definable subsets of U, where $V_k(x)=\{u \in xV : \pi_k(u)=(l_1(x),\dots,l_k(x))\}$ (note that we have $xV=yV$ iff $V_k(x)=V_k(y)$).

The function

$$\alpha_{k+1}: U_k \to N \cup \{+\infty\}, \quad \alpha_{k+1}(x)=\sup \pi^{k+1}(V_k(x))$$

is definable. Note that, if $V_k(x)=V_k(y)$, then $\alpha_{k+1}(x)=\alpha_{k+1}(y)$. Hence, if $M=\{x \in U_k : \alpha_{k+1}(x) \in \pi^{k+1}(V_k(x))\}$, then we can define l_{k+1} on M by $l_{k+1}(x)=\alpha_{k+1}(x)$. Let $U_k'=U_k \setminus M$ and suppose that U_k' is non-empty. By o-minimality, the set F of end points of $\alpha_{k+1}(U_k')$ in $\alpha_{k+1}(U_k)$ is finite. If $F=\emptyset$, then $\alpha_{k+1}(U_k')=\alpha_{k+1}(U_k)$ and hence, $U_k'=\emptyset$. Therefore, F is non-empty. Let $a \in F$. Consider the definable sub-family $X_a=\{V_k(x): x \in U_k \text{ and } \alpha_{k+1}(x)=a\}$ of $\{V_k(x): x \in U_k\}$. Let $x_a \in U_k$ satisfy $\alpha_{k+1}(x_a)=a$ and define $l_{k+1}: \{x \in U_k : V_k(x)=V_k(x_a)\} \to \pi^{k+1}(V_k(x_a))$ by $l_{k+1}(x)=b$ where b is some fixed element of $\pi^{k+1}(V_k(x_a))$. For each $x \in U_k$ such that $\alpha_{k+1}(x)=a$ let $\gamma_a(x)=\inf\{z : b \leqslant z < a, (z,a) \subseteq \pi^{k+1}(V_k(x))\}$. If $V_k(x)=V_k(y)$ then $\gamma_a(x)=\gamma_a(y)$. For $x \in U_k$ with $\alpha_{k+1}(x)=a$ let $K_a(x)=\{z \in O : \alpha_{k+1}(zx) \in (\gamma_a(x),a)\}$ where O is the definable neighbourhood of 1 in U with strong definable choice (see Lemma 2.3). This is a definable family of definable non-empty sets such that if $V_k(x)=V_k(y)$ then $K_a(x)=K_a(y)$. On $\{x \in U_k : \alpha_{k+1}(x)=a\}$ define $l_{k+1}(x)=\alpha_{k+1}(k_a(x)x)$ where $k_a(x)$ is a strong definable choice for $K_a(x)$. And therefore we also get l_{k+1} on $X=\{x \in U_k : \alpha_{k+1}(x) \in F\}$ since X is the disjoint union of definable sets X_a with $a \in F$.

If $X \cup M$ is large in U_k then the claim is proved for $k+1$. Otherwise, we have $\dim(U_k \setminus (X \cup M))=\dim U_k$. Now let $J=\alpha_{k+1}(U_k) \setminus F$. Suppose that J is non-empty. Then J is a finite union of open intervals. Let Y be the definable set of all $x \in U_k$ such that $\alpha_{k+1}(x) \in J$ and α_{k+1} is continuous at x. O-minimality implies that Y is large in $U_k \setminus (X \cup M)$ and so, $Y \cup X \cup M$ is large in U_k. Moreover, if $x \in Y$ and $V_k(y)=V_k(x)$, then $y \in Y$. In fact, let $(z_1,z_2) \subseteq J$ be such that $\alpha_{k+1}(x) \in (z_1,z_2)$ and let D be an open definable neighbourhood of x in U such that $\alpha_{k+1}(D) \subseteq (z_1,z_2)$. Then there is $v \in V$ such that $y=xv$, an open definable neighbourhood of y in U is given by Dv and $\alpha_{k+1}(Dv)=\alpha_{k+1}(D) \subseteq (z_1,z_2)$. Therefore $y \in Y$.

Let A be the definable subset of Y consisting of all $x \in Y$ such that there is a definable open neighbourhood D of x in U with $\alpha_{k+1}(D) \subseteq \{z \in J : \alpha_{k+1}(x) \leqslant z\}$. If $V_k(x)=V_k(y)$ and $x \in A$, then $y=xv$ for some $v \in V$, a definable open neighbourhood of y in U is given by Dv and $\alpha_{k+1}(Dv)=\alpha_{k+1}(D)$. So $y \in A$. Clearly, by o-minimality, $\alpha_{k+1}(A)$ is finite and as before we can construct l_{k+1} on A.

Let $B=Y \setminus A$ and suppose that B is non-empty. Then we have a definable family $\{T(x): x \in B\}$ of definable subsets of O, the definable neighbourhood of 1 in U, with strong definable choice (see Lemma 2.3) given by $T(x)=\{z \in O : \alpha_{k+1}(zx) \in S(x)\}$ where $S(x)=\pi^{k+1}(V_k(x)) \cap \{z \in J : z < \alpha_{k+1}(x)\}$. By construction (and similar properties for Y and A), $S(x)$ is infinite for all x in B. Also if $V_k(x)=V_k(y)$, then $y \in B$, $S(x)=S(y)$ and $T(x)=T(y)$. We now show that $T(x)$ is infinite for all $x \in B$. Let $z' < \alpha_{k+1}(x)$

be such that $(z', \alpha_{k+1}(x)) \subseteq S(x)$. Then by continuity of α_{k+1} (and the fact that $x \in B$) there is a definable open neighbourhood D of x such that $\alpha_{k+1}(D) \cap (z', \alpha_{k+1}(x))$ is infinite. Since $\alpha_{k+1}(Ox \cap D) \cap (z', \alpha_{k+1}(x))$ is infinite (because, otherwise we would have $x \in A$), $T(x)$ is infinite as well.

Since O has strong definable choice, we have a strong definable choice t for the definable family $\{T(x): x \in B\}$ and from this we get l_{k+1} for the definable family $\{V_k(x): x \in B\}$ by setting $l_{k+1}(x) = \alpha_{k+1}(t(x)x)$. Note that if $V_k(x) = V_k(y)$ then $V_k(t(x)x) = V_k(t(y)y)$. Let $U_{k+1} = X \cup Y \cup M$. Then U_{k+1} is large in U_k and the claim is proved for $k+1$. $\square$

3. Definable extensions

3.1. Definable G-modules

Definition 3.1. Let G be a definable group. A *definable G-module* (A, γ) is a G-module such that A is a definable abelian group and the action map $\gamma: G \times A \to A$ is definable. We will often write $\gamma(x)(a)$ for $\gamma(x, a)$. Note that in this way we get a homomorphism $\gamma: G \to \mathrm{Aut}_{\mathcal{N}}(A)$ from G into the group of all definable automorphisms of A.

As usual A is *trivial* if $\gamma(x)(a) = a$ for all $x \in G$ and $a \in A$. And A is *faithful* if $\gamma: G \to \mathrm{Aut}_{\mathcal{N}}(A)$ is injective. A definable G-*submodule* of A is a definable subgroup B of A such that B is invariant under γ (that is, $\gamma(x)(B) \subseteq B$ for all $x \in G$). We then have natural induced definable G-modules $(B, \gamma_{|B})$ and $(A/B, \gamma_{A/B})$. We say that A is *definably irreducible* if it has no proper definable G-submodules. The G-submodule $A^G = \{a \in A:$ for all $x \in G,\ \gamma(x)(a) = a\}$ is always definable.

The next lemma follows from Theorem 2.5 but we include here a direct prove based on descending chain condition (DCC).

Lemma 3.2. *Let (A, γ) be a definable G-module. Then A/A^G is a definable group, $\mathrm{Ker}\,\gamma$ is a normal definable subgroup of G, the quotient $\bar{G} = G/\mathrm{Ker}\,\gamma$ is definable and we have a natural induced faithful definable $\bar{G}$-module $(\bar{\gamma}, A)$.*

Also, if U is a definable group and A is a normal subgroup of U then $C_U(A)$ is a normal definable subgroup of U and $U/C_U(A)$ is definable. In particular, $U/Z(U)$ is definable.

Proof. For each $g \in G$ we have a definable endomorphism $\alpha(g): A \to A$ given by $\alpha(g)(a) = \gamma(g)(a) - a$. By definition $A^G = \bigcap_{g \in G} \mathrm{Ker}\,\alpha(g)$ and so by DCC on definable subgroups there are $g_1, \ldots, g_n \in G$ such that $A^G = \bigcap_{i=1}^{n} \mathrm{Ker}\,\alpha(g_i)$. But then the definable map

$$a \mapsto (\alpha(g_1)(a), \ldots, \alpha(g_n)(a)): A \longrightarrow \alpha(g_1)(A) \times \cdots \times \alpha(g_n)(A)$$

shows that A/A^G is definable.

Let $a \in A$ and consider the definable map $\beta(a): G \to A$ given by $\beta(a)(g) = \gamma(g)(a) - a$. The group $\{g \in G: \beta(a)(g) = 0\}$ is a definable subgroup of G and $\mathrm{Ker}\gamma = \bigcap_{a \in A}\{g \in G:$

$\beta(a)(g)=0\}$. So by DCC on definable subgroups there are $a_1,\ldots,a_n \in A$ such that $\mathrm{Ker}\gamma=\bigcap_{i=1}^{n}\{g\in G:\beta(a_i)(g)=0\}$. The definable map

$$g\mapsto(\beta(a_1)(g),\ldots,\beta(a_n)(g)):G\to\beta(a_1)(G)\times\cdots\times\beta(a_n)(G)$$

shows that $G/\mathrm{Ker}\,\gamma$ is definable.

If U is a definable group and A is a normal subgroup then $C_U(A)=\bigcap_{a\in A}C_U(a)$ and by DCC on definable subgroups there are $a_1,\ldots,a_n\in A$ such that $C_U(A)=\bigcap_{i=1}^{n}C_U(a_i)$ and so $C_U(A)$ is definable (and normal). If for each $a\in A$ we define $ad(a):U\to U$ by $ad(a)(u)=aua^{-1}u^{-1}$ then the definable map

$$u\mapsto(ad(a_1)(u),\ldots,ad(a_n)(u)):U\to ad(a_1)(U)\times\cdots\times ad(a_n)(U)$$

shows that $U/C_U(A)$ is definable. □

3.2. *Group cohomology*

In this subsection we assume that (A,γ) is a definable G-module.

Definition 3.3. For each $n\in\mathbb{N}$ let $C^n_{\mathcal{N}}(G,A,\gamma)$ denote the abelian group of all definable functions from G^n into A with pointwise addition. An element of $C^n_{\mathcal{N}}(G,A,\gamma)$ is called a *definable n-cochain (over $\mathcal{N}$)*.

Definition 3.4. The *co-boundary map* $\delta:C^n_{\mathcal{N}}(G,A,\gamma)\to C^{n+1}_{\mathcal{N}}(G,A,\gamma)$, is defined by

$$\delta(c)(g_1,\ldots,g_{n+1})=\gamma(g_1)(c(g_2,\ldots,g_{n+1}))+\sum_{i=1}^{n}(-1)^i c(g_1,\ldots,g_i\cdot g_{i+1},\ldots,g_{n+1})$$
$$+(-1)^{n+1}c(g_1,\ldots,g_n).$$

It is clear that $\delta(c)$ is also definable.

Lemma 3.5. $\delta\circ\delta=0$.

Proof. This is a simple calculation. □

Definition 3.6. We therefore have a *complex* $C^*_{\mathcal{N}}(G,A,\gamma)$. Let $B^n_{\mathcal{N}}(G,A,\gamma)$ denote the image of $\delta:C^{n-1}_{\mathcal{N}}(G,A,\gamma)\to C^n_{\mathcal{N}}(G,A,\gamma)$ and let $Z^n_{\mathcal{N}}(G,A,\gamma)$ denote the kernel of $\delta:C^n_{\mathcal{N}}(G,A,\gamma)\to C^{n+1}_{\mathcal{N}}(G,A,\gamma)$. The *n-cohomology group over* $\mathcal{N}$ $H^n_{\mathcal{N}}(G,A,\gamma)$ is the abelian group $Z^n_{\mathcal{N}}(G,A,\gamma)/B^n_{\mathcal{N}}(G,A,\gamma)$. The elements of $B^n_{\mathcal{N}}(G,A,\gamma)$ are the *definable n-coboundaries* and the elements of $Z^n_{\mathcal{N}}(G,A,\gamma)$ are the *definable n-cocycles*.

Remark 3.7. Let (A,γ) be a definable G-module. Suppose that $A=A_1\times A_2$ and that A_1 and A_2 are invariant under the action of G on A. Then $H^n_{\mathcal{N}}(G,A,\gamma)$ is isomorphic with $H^n_{\mathcal{N}}(G,A_1,\gamma_{|A_1})\times H^n_{\mathcal{N}}(G,A_2,\gamma_{|A_2})$.

3.3. Definable extensions

Definition 3.8. Let U be a definable group. (U,i,j) is a *definable extension* of G by A if we have an exact sequence

$$1 \to A \xrightarrow{i} U \xrightarrow{i} G \to 1$$

in the category of definable groups with definable homomorphisms. If (U,i,j) is a *definable extension* of G by A and U is abelian, we say that (U,i,j) is a *definable abelian extension* of G by A. A *definable section* is a definable map $s: G \to U$ such that $j(s(g)) = g$ for all $g \in G$.

Definition 3.9. Two definable extensions $1 \to A \xrightarrow{i} U \xrightarrow{j} G \to 1$ and $1 \to A \xrightarrow{i'} U' \xrightarrow{j'} G \to 1$ are *definably equivalent* if there is a definable homomorphism $\varphi: U \to U'$ such that

$$\begin{tikzcd}
 & & U \arrow[dr, "j"] \arrow[dd, "\varphi"] & & \\
1 \arrow[r] & A \arrow[ur, "i"] \arrow[dr, "i'"'] & & G \arrow[r] & 1 \\
 & & U' \arrow[ur, "j'"'] & &
\end{tikzcd}$$

is a commutative diagram.

Note. Below we will sometimes assume that $A \trianglelefteq U$ and write (U,j) for (U,i,j).

Theorem 3.10. *Let* $1 \to A \to U \xrightarrow{j} G \to 1$ *be a definable extension. Then there is a definable section* $s: G \to U$.

Proof. Let $l: U \to l(U) \subseteq U$ be a strong definable choice given by Theorem 2.5 for the definable family $\{xA : x \in U\}$. Since the definable family $\{j^{-1}(g) : g \in G\}$ is the same as the definable family $\{xA : x \in U\}$ we can define $s: G \to U$ by $s(g) = l(x)$ for some (equivalently, for all) $x \in j^{-1}(g)$. $\square$

Corollary 3.11. *If* V *is a normal definable subgroup of a definable group* U, *then there s a definable extension* $1 \to V \to U \xrightarrow{l} U/V \to 1$ *with a definable section* $s: U/V \to U$.

Proof. This follows from Theorem 3.10 since, by Theorem 2.5, there is a strong definable choice $l: U \to U/V$ for the definable family $\{xV : x \in U\}$. $\square$

Remark 3.12. Suppose that we have a definable extension $1 \to A \to U \xrightarrow{l} G \to 1$ and $B \trianglelefteq G$ is definable. Then $C = l^{-1}(B) \trianglelefteq U$ and $A \trianglelefteq C$. Moreover, suppose that we have a definable extension $1 \to B \to G \xrightarrow{j} H \to 1$. Then we have definable extensions $1 \to C \to U \xrightarrow{j \circ l} H \to 1$ and $1 \to A \to C \xrightarrow{l_{|C}} B \to 1$.

Remark 3.13. Suppose that we have a definable extension $1 \to A \to U \xrightarrow{l} G \to 1$ and $A \trianglelefteq V \trianglelefteq U$ is a definable normal subgroup. Then we have definable extensions $1 \to A \to V \xrightarrow{l_{|V}} H \to 1$, $1 \to V \to U \xrightarrow{k} U/V \to 1$ and $1 \to H \to G \xrightarrow{p} U/V \to 1$ such that $p \circ l = k$.

The lemmas we prove below will be very useful later on. These results are about the invariance of notions such as definably compact, definably-connected and with no definably compact parts under definable extensions.

Lemma 3.14. *Let $1 \to A \to U \xrightarrow{j} G \to 1$ be a definable extension. Then U is definably compact if and only if A and G are definably compact.*

Proof. Suppose that U is definably compact. Then since A is a closed definable subgroup of U it must be that A is definably compact. We now show that G is also definably compact. Let $s : G \to U$ be a definable section and let $\alpha : (a,b) \subseteq N \to G$ be a definable continuous map where $-\infty \leqslant a < b \leqslant +\infty$. Let $\beta : (a,b) \to U$ be the definable map given by $\beta(x) = s(\alpha(x))$. Since U is definably compact, the limit $\lim_{x \to a^+} \beta(x)$ (resp., $\lim_{x \to b^-} \beta(x)$) exists in U. Since j is continuous and $\alpha = j \circ \beta$, the limit $\lim_{x \to a^+} \alpha(x)$ (resp., $\lim_{x \to b^-} \alpha(x)$) exists in G and G is definably compact.

Suppose now that A and G are definably compact. Let $s : G \to U$ be a definable section and let $\alpha : (a,b) \subseteq N \to U$ be a definable continuous map where $-\infty \leqslant a < b \leqslant +\infty$. Let $\beta : (a,b) \to G$ be the definable map given by $\beta(x) = j(\alpha(x))$ and let $\gamma : (a,b) \to U$ be the definable map given by $\gamma(x) = s(\beta(x))$. Then there is a definable map $\delta : (a,b) \subseteq N \to A$ such that, for all $x \in (a,b)$, we have $\alpha(x) = \delta(x)\gamma(x)$. Since A is definably compact, the limit $\lim_{x \to a^+} \delta(x)$ (resp., $\lim_{x \to b^-} \delta(x)$) exists in A. Therefore, to show that the limit $\lim_{x \to a^+} \alpha(x)$ (resp., $\lim_{x \to b^-} \alpha(x)$) exists in U, i.e, to show that U is definably compact, it remains to show that the limit $\lim_{x \to a^+} \gamma(x)$ (resp., $\lim_{x \to b^-} \gamma(x)$) exists in U.

Since G is definably compact, the limit $g = \lim_{x \to a^+} \beta(x)$ (resp., $g = \lim_{x \to b^-} \beta(x)$) exists in G. By o-minimality, $s : G \to U$ is continuous on a large definable subset of G and so there is $h \in G$ such that $s : G \to U$ is continuous at hg. Let $\gamma' : (a,b) \subseteq N \to U$ be the definable map given by $\gamma'(x) = s(h\beta(x))$. Then by the continuity of s at hg, the limit $\lim_{x \to a^+} \gamma'(x)$ (resp., $\lim_{x \to b^-} \gamma'(x)$) exists in U. Note that, for all $x \in (a,b)$, we have $\gamma'(x)(\gamma(x))^{-1} \in j^{-1}(h)$ and $j^{-1}(h)$ is definably compact (because A is definably compact and $j^{-1}(h)$ is definably homeomorphic to A). Therefore, the limit $\lim_{x \to a^+} \sigma(x)$ (resp., $\lim_{x \to b^-} \sigma(x)$) where for all $x \in (a,b)$ we set $\sigma(x) = (\gamma'(x)(\gamma(x))^{-1})^{-1}$, exists in U. But $\gamma(x) = \sigma(x)(\gamma'(x))^{-1}$ and so the limit $\lim_{x \to a^+} \gamma(x)$ (resp., $\lim_{x \to b^-} \gamma(x)$) exists in U. $\square$

Lemma 3.15. *Let $1 \to A \to U \xrightarrow{j} G \to 1$ be a definable extension. If U is definably-connected then G is definably-connected. Moreover, when A is definably-connected, then U is definably-connected if and only if G is definably-connected.*

Proof. Since j is a continuous and surjective definable map, if U is definably-connected so is G. Therefore, it remains to show that if A and G are definably-connected, U is also definably-connected. So suppose that A and G are definably-connected but the definably-connected component U^0 of U is a proper definable normal subgroup of U. Since $\dim U = \dim A + \dim G$, $j(U^0)$ is a definably-connected definable normal subgroup of G with the same dimension as G. Therefore, because G is definably-connected, we have $j(U^0) = G$. On the other hand, since A is definably-connected, for each $g \in G$, the fibre $j^{-1}(g)$ is also definably-connected and hence $j^{-1}(g) \subseteq U^0$. But this implies that $U \subseteq U^0$. $\square$

Definition 3.16. Let $\mathscr{I}$ be a definable o-minimal expansion of an ordered group $(I, 0, +, <)$. We say that an $\mathscr{I}$-definable abelian group U is *globally over I* if there are $\mathscr{I}$-definable subgroups $1 = U_0 < U_1 < \cdots < U_n = U$ such that, for each $j \in \{1, \dots, n\}$, the group U_j/U_{j-1} is $\mathscr{I}$-definably isomorphic to an $\mathscr{I}$-definable group with domain I and identity 0. We say that an $\mathscr{I}$-definable solvable group U is *globally over I* if there are $\mathscr{I}$-definable subgroups $1 = U_0 \trianglelefteq U_1 \trianglelefteq \cdots \trianglelefteq U_n = U$ such that, for each $j \in \{1, \dots, n\}$, the group U_j/U_{j-1} is an $\mathscr{I}$-definable abelian group globally over I.

Note that if an $\mathscr{I}$-definable solvable group U is globally over I, then U has no $\mathscr{I}$-definably compact parts.

Lemma 3.17. *Let $\mathscr{I}$ be a definable o-minimal expansion of an ordered group $(I, 0, +, <)$ and let $1 \to A \to U \to G \to 1$ be an $\mathscr{I}$-definable extension of $\mathscr{I}$-definable solvable groups with $\dim A, \dim G \geqslant 1$. Then U is globally over I iff both A and G are globally over I.*

Proof. Suppose that A and G are $\mathscr{I}$-definable groups globally over I. Let $1 = A_0 \trianglelefteq A_1 \trianglelefteq \cdots \trianglelefteq A_r = A$ (resp., $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_s = G$) show that A (resp., G) is globally over I. For $i = 1, \dots, r$ set $U_i = A_i$ and for $i = r+1, \dots, r+s$ set U_i to be the $\mathscr{I}$-definable subgroup of U such that $U_i/U_{i-1} = G_{i-r}$. Then $1 = U_0 \trianglelefteq U_1 \trianglelefteq \cdots \trianglelefteq U_{r+s} = U$ shows that U is an $\mathscr{I}$-definable group globally over I.

Suppose that U is an $\mathscr{I}$-definable group globally over I and let $1 = U_0 \trianglelefteq U_1 \trianglelefteq \cdots \trianglelefteq U_n = U$ witness this fact. We now show that the result holds for A. Let $A_i = A \cap U_i$ for each $i = 1, \dots, n$ so that we have $1 = A_0 \trianglelefteq A_1 \trianglelefteq \cdots \trianglelefteq A_n = A$. Since $A \cap U_{i-1} = (A \cap U_i) \cap U_{i-1}$ we have $\mathscr{I}$-definable isomorphisms $A_i/A_{i-1} = A \cap U_i/A \cap U_{i-1} \simeq (A \cap U_i)U_{i-1}/U_{i-1}$. Let $\alpha : U_i \to U_i/U_{i-1}$ be the natural $\mathscr{I}$-definable homomorphism; then $(A \cap U_i)U_{i-1}/U_{i-1} = \alpha(A \cap U_i) \trianglelefteq \alpha(U_i) = U_i/U_{i-1}$, and hence A_i/A_{i-1} is $\mathscr{I}$-definably isomorphic to a normal $\mathscr{I}$-definable subgroup of U_i/U_{i-1}. Suppose that U is abelian. Then, either $A_i = A_{i-1}$ or $A_i/A_{i-1} \simeq U_i/U_{i-1}$; thus the result holds for A. In general, the result holds for A since U_n/U_{n-1} is abelian and of the same form as U.

We now prove the result for G. Let $G_i = U_iA/A$ for each $i = 1, \dots, n$ so that we have $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$. Since $U_iA = U_i(U_{i-1}A)$, we have $\mathscr{I}$-definable isomorphisms $G_i/G_{i-1} \simeq U_iA/U_{i-1}A \simeq U_i/U_i \cap U_{i-1}A$. On the other hand, we have an $\mathscr{I}$-definable extension $1 \to U_i \cap U_{i-1}A/U_{i-1} \to U_i/U_{i-1} \to U_i/U_i \cap U_{i-1}A \to 1$. Therefore, we have an $\mathscr{I}$-definable extension $1 \to U_i \cap U_{i-1}A/U_{i-1} \to U_i/U_{i-1} \to G_i/G_{i-1} \to 1$. Suppose that U is abelian. Then, either $G_i = G_{i-1}$ or $G_i/G_{i-1} \simeq U_i/U_{i-1}$;

thus the result holds for G. In general, the result holds for G because of this $\mathscr{I}$-definable extension and since U_n/U_{n-1} is abelian and of the same form as U. □

The same purely algebraic argument used in the proof of Lemma 3.17 shows the following result.

Lemma 3.18. *Let $\mathscr{I}$ be a definable o-minimal structure and let $1 \to A \to U \to G \to 1$ be an $\mathscr{I}$-definable extension of $\mathscr{I}$-definable solvable groups with $\dim A, \dim G \geqslant 1$. Then U is an $\mathscr{I}$-definable group with no $\mathscr{I}$-definably compact parts iff A and G are $\mathscr{I}$-definable groups with no $\mathscr{I}$-definably compact parts.*

3.4. Definable G-kernels

Notation. Let A be a definable group. $\mathrm{Aut}_{\mathscr{N}}(A)$ denotes the group of all definable automorphisms of A, $\mathrm{Inn}(A)$ the group of all inner automorphisms of A and $\mathrm{Out}_{\mathscr{N}}(A) = \mathrm{Aut}_{\mathscr{N}}(A)/\mathrm{Inn}(A)$. Let $\imath : \mathrm{Aut}_{\mathscr{N}}(A) \to \mathrm{Out}_{\mathscr{N}}(A)$ denote the natural homomorphism. If $A \trianglelefteq U$ and $u \in U$ then we denote by $\langle u \rangle$ the automorphism of A given by $\langle u \rangle(a) = uau^{-1}$ for all $a \in A$.

Definition 3.19. Let G be a definable group. A *definable G-kernel* (A, θ) is a definable group A with a homomorphism $\theta : G \to \mathrm{Out}_{\mathscr{N}}(A)$ such that there is a homomorphism $\alpha : G \to \mathrm{Aut}_{\mathscr{N}}(A)$ such that: (i) $\theta = \imath(\alpha)$; (ii) the map $\alpha : G \times A \to A$, $\alpha(x, a) = \alpha(x)(a)$ is definable and (iii) there is a definable function $h_\alpha : G \times G \to A$ such that, for all $x, y \in G$, we have $h_\alpha(x, 1) = h_\alpha(1, y) = 1$ and

$$\forall x, y \in G, \ \alpha(x)\alpha(y) = \langle h_\alpha(x, y) \rangle \alpha(xy). \tag{1}$$

Note that θ induces a definable action $\theta_0 : G \times Z(A) \to Z(A)$ making the centre $Z(A)$ of A a *definable G-module*. We say that α as above is a *definable representative* of the definable G-kernel (A, θ) and we write $\alpha \in \theta$.

Definition 3.20. Let G be a definable group and B an abelian definable group. Two definable G-kernels (A_i, θ_i) with $i = 1, 2$ with centre B, that is $Z(A_1) = Z(A_2) = B$, are *definably equivalent* if there is a definable isomorphism $\sigma : A_1 \to A_2$ and there are $\alpha_i \in \theta_i$ for $i = 1, 2$, such that for all $b \in B$, $\sigma(b) = b$ and for each $x \in G$, there is $i_x \in \mathrm{Inn}(A_2)$ such that $\sigma\alpha_1(x)\sigma^{-1} = i_x\alpha_2(x)$. This relation is an equivalence relation and the set of all the classes is denoted by $K_{\mathscr{N}}(G, B)$.

Remark 3.21. Let (U, j) be a definable extension of G by A. Then there is a canonical homomorphism $\theta_U : G \to \mathrm{Out}_{\mathscr{N}}(A)$ such that (A, θ_U) is a definable G-kernel: for each $x \in G$ take $\theta_U(x) = \{\langle u \rangle : u \in j^{-1}(x)\}$ with definable representative given by

$$\alpha_{U,s} : G \to \mathrm{Aut}_{\mathscr{N}}(A), \ \alpha_{U,s}(x)(a) = \langle s(x) \rangle(a)$$

and $h_{\alpha_{U,s}}(x, y) = s(x)s(y)s(xy)^{-1}$ where $s : G \to U$ is a definable section. Using the fact that for all $x, y, z \in G$, by associativity, the product $\alpha(x)\alpha(y)\alpha(z)$ may be calculated in two different ways, a simple calculation shows that

$$\alpha_{U,s}(x)(h_{\alpha_{U,s}}(y,z))h_{\alpha_{U,s}}(x,yz) = h_{\alpha_{U,s}}(x,y)h_{\alpha_{U,s}}(xy,z). \tag{2}$$

If $s' : G \to U$ is another definable section, then there is a definable function $k_{s,s'} : G \to A$ given by $s'(x) = k_{s,s'}(x)s(x)$ for all $x \in G$, and we have $\alpha_{U,s'}(x) = < k_{s,s'}(x) > \alpha_{U,s}(x)$ for all $x \in G$.

Definition 3.22. A definable G-kernel (A,θ) is *definably extendible* if there is a definable extension (U,j) of G by A such that (A,θ_U) is definably equivalent to (A,θ). We say in this case that (U,j) is *definably compatible* with the G-kernel. We denote by $\mathrm{Ext}_{\mathcal{N}}(G,A,\theta)$ the set of all equivalence classes of definable extensions of G by A definably compatible with the G-kernel (A,θ). Let $EK_{\mathcal{N}}(G,B)$ be the subset of $K_{\mathcal{N}}(G,B)$ of all classes (A,θ) such that $\mathrm{Ext}_{\mathcal{N}}(G,A,\theta)$ is non-empty. Note that $EK_{\mathcal{N}}(G,B)$ is a well defined subset of $K_{\mathcal{N}}(G,B)$.

3.5. Existence of definable extensions

With the set up we have established the proofs of results of this subsection are as in the classical case. For details see the proofs of the corresponding results in [5] and [6] respectively. We will include here only the constructions that will be useful later.

Proposition 3.23. *Let* $(A,\theta) \in EK_{\mathcal{N}}(G,B)$, $(U,j) \in \mathrm{Ext}_{\mathcal{N}}(G,A,\theta)$ *and* $s : G \to U$ *a definable section. Then there is* $(V_s,i_s,j_s) \in \mathrm{Ext}_{\mathcal{N}}(G,A,\theta)$ *with domain* $A \times G$ *and multiplication given by*

$$\forall a,b \in A \forall x,y \in G,\ (a,x)(b,y) = (a[\alpha_{U,s}(x)(b)]h_{\alpha_{U,s}}(x,y),xy) \tag{3}$$

which is canonically definably isomorphic with U.

Proof. From Eq. (2) we see that V_s is a definable group with identity element $(1,1)$. The inverse of (a,x) is $(\alpha_{U,s}(x)^{-1}[h_{\alpha_{U,s}}(x,x^{-1})a]^{-1},x^{-1})$. The definable homomorphism $i_s : A \to V_s$ is given by $i_s(a) = (a,1)$ and $j_s : V_s \to G$ by $j_s((a,x)) = x$. The map $t_s : G \to V_s$ defined by $t_s(x) = (1,x)$ is a definable section and for all $x \in G$ we have $\langle t_s(x)\rangle = \alpha_{U,s}(x)$. Therefore, $(V_s,i_s,j_s) \in \mathrm{Ext}_{\mathcal{N}}(G,A,\theta)$. Also, the map $as(x) \mapsto (a,x) : U \to V_s$ is a definable isomorphism. □

Note that, if $s' : G \to U$ is another definable section and $(V_{s'},i_{s'},j_{s'}) \in \mathrm{Ext}_{\mathcal{N}}(G,A,\theta)$ the corresponding definable extension given by Proposition 3.23, then there is a definable function $k_{s,s'} : G \to A$ given by $s'(x) = k_{s,s'}(x)s(x)$ such that

$$\forall x,y \in G,\ h_{\alpha_{U,s'}}(x,y) = k_{s,s'}(x)\alpha_{U,s}(x)(k_{s,s'}(y))h_{\alpha_{U,s}}(x,y)k_{s,s'}(xy)^{-1} \tag{4}$$

and the map $(a,x) \mapsto (ak_{s,s'}(x)^{-1},x) : V_s \to V_{s'}$ is a definable isomorphism.

Proposition 3.24. *With the assumptions of Proposition* 3.23, U *is definably isomorphic with* $A \rtimes_\gamma G$ *for some homomorphism* $\gamma : G \to \mathrm{Aut}_{\mathcal{N}}(A)$ *such that the induced map* $\gamma : G \times A \to A$ *is definable iff there is a definable function* $g : G \to A$ *such that*

$$\forall x,y \in G,\ h_{\alpha_{U,s}}(x,y) = \alpha_{U,s}(x)(g(y))g(x)g(xy)^{-1}. \tag{5}$$

Proof. If $g:G\to A$ is a definable map satisfying Eq. (5), then the definable map $x\mapsto (g(x)^{-1},x):G\to V_s$ is a definable injective homomorphism. $\square$

The following remark, which we will not use in this paper, is proved as its classical analogue (see [6, Theorem 11.1]).

Remark 3.25. Let $(A,\theta)\in EK_{\mathcal{N}}(G,B)$ and $(U,j)\in \mathrm{Ext}_{\mathcal{N}}(G,A,\theta)$. Then there is a canonical bijection from $\mathrm{Ext}_{\mathcal{N}}(G,A,\theta)$ into $H^2_{\mathcal{N}}(G,B,\theta_0)$ sending (U,j) into the identity of $H^2_{\mathcal{N}}(G,B,\theta_0)$.

Remark 3.26. Let $(A,\theta)\in EK_{\mathcal{N}}(G,B)$ and $(U,j)\in \mathrm{Ext}_{\mathcal{N}}(G,A,\theta)$. Suppose that $A=A_1\times A_2$, $B=B_1\times B_2$ and $\mathrm{Aut}_{\mathcal{N}}\ (A)=\mathrm{Aut}_{\mathcal{N}}(A_1)\times \mathrm{Aut}_{\mathcal{N}}(A_2)$. Then $\theta=(\theta_1,\theta_2):G\to \mathrm{Out}_{\mathcal{N}}(A_1)\times \mathrm{Out}_{\mathcal{N}}(A_2)$ and, for each $i=1,2$, we have $(A_i,\theta_i)\in K_{\mathcal{N}}(G,B_i)$.

Let θ_U be as in Remark 3.21 and for $i=1,2$ let θ^i_U be such that $\theta^i_U(x)=\theta_U(x)_{|A_i}$ for all $x\in G$. Then $(A,\theta)=(A,\theta_U)$ in $K_{\mathcal{N}}(G,B)$ and $(A_i,\theta_i)=(A_i,\theta^i_U)$ in $K_{\mathcal{N}}(G,B_i)$. Since $A_1\trianglelefteq U$, we have definable extensions

$$1\to A\ \to\ U\xrightarrow{j} G\to 1,$$

$$1\to A_2\to U_2\xrightarrow{j_2} G\to 1,$$

$$1\to A_1\to U\xrightarrow{l_2} U_2\to 1$$

such that $j_2\circ l_2=j$ and $l_{2|A_2}=1_{A_2}$. If $s:G\to U$ is a definable section, then $s_2=l_2\circ s:G\to U_2$ and $t_2=s\circ j_2:U_2\to U$ are definable sections. Clearly $(A_2,\theta_2)=(A_2,\theta_{U_2})$ in $K_{\mathcal{N}}(G,B_2)$, also $(U_2,j_2)\in \mathrm{Ext}_{\mathcal{N}}(G,A_2,\theta_2)$ and $(A_2,\theta_2)\in EK_{\mathcal{N}}(G,B_2)$. Similarly, we have $(A_1,\theta_1)=(A_1,\theta_{U_1})$ in $K_{\mathcal{N}}(G,B_1)$, also $(U_1,j_1)\in \mathrm{Ext}_{\mathcal{N}}(G,A_1,\theta_1)$ and $(A_1,\theta_1)\in EK_{\mathcal{N}}(G,B_1)$. We have definable sections $s_1=l_1\circ s:G\to U_1$ and $t_1=s\circ j_1: U_1\to U$.

Using the notation of Remark 3.21, it is easy to see that $\alpha_{U,s}(x)=(\alpha_{U_1,s_1}(x),\alpha_{U_2,s_2}(x))$. Secondly it follows that we have $\alpha_{U,t_2}(x)=\alpha_{U,s}(j_2(x))=\alpha_{U_1,s_1}(j_2(x))$. Thirdly we have $h_{\alpha_{U,s}}(x,y)=(h_{\alpha_{U_1,s_1}}(x,y),h_{\alpha_{U_2,s_2}}(x,y))$ and $h_{\alpha_{U,t_2}}(x,y)=h_{\alpha_{U_1,s_1}}(j_2(x),j_2(y))$.

Therefore, by Proposition 3.24, U is definably isomorphic with $A_1\rtimes_\mu U_2$ for some $\mu:U_2\to \mathrm{Aut}_{\mathcal{N}}(A_1)$ such that the map $\mu:U_2\times A_1\to A_1$ given by $\mu(u,a)=\mu(u)(a)$ is definable iff there if a definable map $g:G\to A_1$ such that for all $x,y\in G$, $h_{\alpha_{U_1,s_1}}(x,y)=\alpha_{U_1,s_1}(x)(g(y))g(x)g(xy)^{-1}$.

Proposition 3.27. *Suppose that $(U,j)\in \mathrm{Ext}_{\mathcal{N}}(G,A,\gamma)$. Then there is a definable 2-cocycle $c\in Z^2_{\mathcal{N}}(G,A,\gamma)$ associated with (U,j) which is unique in $H^2_{\mathcal{N}}(G,A,\gamma)$. Therefore, (in A) we have*

$$\forall g,h,k\in G,\ \ \gamma(g)(c(h,k))-c(gh,k)+c(g,hk)-c(g,h)=0. \tag{6}$$

Moreover, there is $(V,i,l)\in \mathrm{Ext}_{\mathcal{N}}(G,A,\gamma)$ with domain $A\times G$ and multiplication given by

$$\forall a,b\in A,\ \ \forall g,h\in G,\ \ (a,g)(b,h)=(a+\gamma(g)(b)+c(g,h),gh) \tag{7}$$

which is canonically definably isomorphic to (U,j).

Proof. Let $s:G \to U$ be a definable section and define $c(g,h)=s(g)s(h)s(gh)^{-1}$ for all $g,h \in G$. Then, from equation (6) V is a group with identity $(-c(1,1),1)$. The definable homomorphism $i:A \to V$ is given by $i(a)=(a-c(1,1),1)$ and $l:V \to G$ by $l(a,g)=g$. The map $as(g) \mapsto (a,g):U \to V$ is a definable isomorphism.

If $s':G \to U$ is another definable section and $c' \in Z^2_{\mathcal{N}}(G,A,\gamma)$ is the corresponding definable 2-cocycle and $(V',i',l') \in \mathrm{Ext}_{\mathcal{N}}(G,A,\gamma)$ is the corresponding definable extension, then there is a definable function $b:G \to A$ given by $s'(g)=b(g)s(g)$ such that

$$\forall g,h \in G,\ c'(g,h)-c(g,h)=\gamma(g)(b(h))-b(gh)+b(g), \tag{8}$$

i.e., c and c' determine the same element in $H^2_{\mathcal{N}}(G,A,\gamma)$ and the map $(a,g) \mapsto (a-b(g),g):V \to V'$ is a definable isomorphism. □

Proposition 3.28. *With the assumptions of Proposition* 3.27, *U is definably isomorphic with $A \rtimes_\gamma G$ iff there is a definable function $a:G \to A$ such that*

$$\forall g,h \in G,\ c(g,h)=\gamma(g)(a(h))-a(gh)+a(g). \tag{9}$$

Proof. If $a:A \to G$ exists and satisfies Eq. (9), then $g \mapsto (-a(g),g):G \to V$ is a definable injective homomorphism. □

The following remark, which we will not use in this paper, is proved as its classical analogue (see [5, (3.2)]).

Remark 3.29. Let (A,γ) be a definable G-module. Then there is a bijection from $\mathrm{Ext}_{\mathcal{N}}(G,A,\gamma)$ onto $H^2_{\mathcal{N}}(G,A,\gamma)$ sending the class of $A \rtimes_\gamma G$ into the identity of $H^2_{\mathcal{N}}(G,A,\gamma)$.

Remark 3.30. As in [6, Sections 4 and 5], the set $K_{\mathcal{N}}(G,B)$ can be made into an abelian group such that $EK_{\mathcal{N}}(G,B)$ is a subgroup of $K_{\mathcal{N}}(G,B)$. The map of Remark 3.29 is an isomorphism between $H^2_{\mathcal{N}}(G,B,\theta_0)$ and $\mathrm{Ext}_{\mathcal{N}}(G,B,\theta_0)$ (see [5, (3.2)]). Moreover, as in [6, Theorem 11.1], the map of Remark 3.25 from $H^2_{\mathcal{N}}(G,B,\theta_0)$ into $\mathrm{Ext}_{\mathcal{N}}(G,A,\theta)$ for a fixed (U,j) in $\mathrm{Ext}_{\mathcal{N}}(G,B,\theta)$ is the composition of the isomorphism from $H^2_{\mathcal{N}}(G,B,\theta_0)$ into $\mathrm{Ext}_{\mathcal{N}}(G,B,\theta_0)$ and the map from $\mathrm{Ext}_{\mathcal{N}}(G,B,\theta_0)$ into $\mathrm{Ext}_{\mathcal{N}}(G,A,\theta)$ which sends (V,l) into its left translation by (U,j).

4. Definably compact definable groups

In this section we prove that a definably compact definable solvable group is abelian-by-finite. This will follow after we show that a definably-connected, definably compact, definable G-module where G is infinite and definably-connected is trivial. Before we proceed, we need the following easy lemma.

Lemma 4.1. *Let U be an infinite definable group and let V be a definable subgroup such that* $\dim V < \dim U$. *Then there is a definable continuous embedding $\sigma:(a,b) \to U$ such that* $\lim_{t\to a^+}\sigma(t) = 1$ *and* $\sigma(a,b) \subseteq U \setminus V$.

Proof. Let (O,ϕ) be a definable chart of 1 (the identity of U). Then $\phi(O)$ is a definable open subset of N^n where $n = \dim U$. By Lemma 2.3 and Fact 2.2, we may assume, without loss of generality, that O has strong definable choice. Let $e=\phi(1)$ and let $B \subseteq \phi(O)$ be a closed box containing e. Then by Fact 2.2, B has strong definable choice.

Let $D = \phi(V \cap O) \cap B$. Then since $\dim V < \dim U$, it follows that $\dim D < n$ and $\dim(B \setminus D) = n$. Let $\mathscr{C}$ be a cell decomposition of B compatible with D and $B \setminus D$. Then there are $C, C' \in \mathscr{C}$ such that $\dim C = \dim B$, $C' \subseteq D \subseteq B$ and the closure $\bar{C'}$ of C' is in the closure $\bar{C}$ of C in B. Note that since V is closed in U, we have that D is closed in B and there are inclusions $\bar{C'} \subseteq D$ and $C \subseteq B \setminus D$.

We now show by induction on $\dim B$ that there is a definable continuous embedding $\alpha:(a,b) \to C$ such that $\lim_{t\to a^+}\alpha(t) \in \bar{C'}$. If $\dim B = 1$, then the result is clear. So suppose that $\dim B > 1$ and the result holds for lower dimensions. Let $\pi: N^n \to N^{n-1}$ be the projection onto the first $n-1$-coordinates. Then $\pi(B)$ is a closed box of dimension $\dim B - 1$ with strong definable choice by Fact 2.2. The projection $\pi(\mathscr{C})$ is a cell decomposition of $\pi(B)$, $\pi(C), \pi(C')$ are cells of $\pi(\mathscr{C})$ such that $\dim \pi(C) = \dim \pi(B)$ and $\overline{\pi(C')}$ is in the closure $\overline{\pi(C)}$ of $\pi(C)$ in $\pi(B)$. By the induction hypothesis, there is a definable continuous embedding $\beta:(a_1,b_1) \to \pi(C)$ such that $\lim_{x\to a_1^+}\beta(x) \in \overline{\pi(C')}$. Let $\{T(x): x\in(a_1,b_1)\}$ be the definable family of non-empty definable subsets of C given by $T(x)=\{c\in C: \pi(c)=\beta(x)\}$. Let $\alpha_1:(a_1,b_1) \to C$ be a strong definable choice for $\{T(x): x\in(a_1,b_1)\}$. By o-minimality, there is $b\in(a_1,b_1)$ such that $\alpha=\alpha_{1|(a,b)}$ where $a = a_1$ is a definable continuous embedding. This α satisfies the claim for B.

To finish the proof of the lemma, take $\sigma:(a,b) \to U$ given by $\sigma(t)=(\phi^{-1}\circ\alpha)(t)v^{-1}$ where $\phi(v) = \lim_{t\to a^+}\alpha(t) \in D$. $\square$

Lemma 4.2. *Let (A,γ) be a non-trivial definable G-module, where A and G are infinite definably-connected definable groups. Then there are an infinite minimal definable subgroup B of A/A^G, a definable G-submodule H of A and a definable family $\Gamma: G\times B \to H$ of definable homomorphisms from B into H such that $\Gamma(1,b) = 0 = \Gamma(g,0)$ for all $b\in B$ and $g\in G$. Moreover, for suitable Γ,* $\mathrm{Ker}_G\,\Gamma=\{g\in G: \textit{for all } b\in B,\ \Gamma(g,b)=0\}$ *is a proper definable subgroup of G and for all $g\in G$ and $c\in B$, there is $d\in B$ such that $\Gamma(-,c): G\to H$ is continuous at g and $\Gamma: G\times B\to H$ is continuous at $(g,c+d)$.*

Proof. Since $A^G \neq A$ and A is definably-connected, there is an infinite minimal definable subgroup B of A/A^G. By Remark 3.12, there is a definable subgroup C of A containing A^G and such that $C/A^G = B$. Let H be the smallest definable G-submodule of A containing C. Since C is a definable extension of B by A^G, there is a definable section $s: B\to C$. Let $c(b,b')=s(b)+s(b')-s(b+b')$ be the corresponding definable 2-cocycle.

Let $\Gamma: G\times B\to H$ be the definable map given by $\Gamma(g,b)=\gamma(g)(s(b))-s(b)$ for all $b\in B$ and $g\in G$. Then clearly $\Gamma(1,b) = 0 = \Gamma(g,0)$ for all $b\in B$ and $g\in G$. We now show, for each $g\in G$, that $\Gamma(g)=\Gamma(g,-):B\to H$ is a homomorphism. Let $b,b'\in B$. Then

$$\begin{aligned}&\Gamma(g)(b+b')-\Gamma(g)(b)-\Gamma(g)(b')\\&\quad=\gamma(g)(s(b+b'))-\gamma(g)(s(b))-\gamma(g)(s(b'))-s(b+b')+s(b)+s(b')\\&\quad=-c(b,b')+c(b,b')\end{aligned}$$

$(\gamma(g)(s(b+b'))-\gamma(g)(s(b'))-\gamma(g)(s(b))=-c(b,b')$ since $c(b,b')\in A^G$). Adding to this last equation, the equation for the 2-cocycle, we get $\Gamma(g)(b+b')-\Gamma(g)(b')-\Gamma(g)(b)=0$. So $\Gamma(g)$ is a definable homomorphism.

Clearly, $\mathrm{Ker}_G\,\Gamma$ is a definable subgroup of G. We show that it is proper. Suppose not. Then $\gamma(g)(s(b))=s(b)$ for all $g\in G$ and for all $b\in B$. Let $c\in C$. Then $c=a+s(b)$ for some $a\in A^G$ and $b\in B$. Therefore, $\gamma(g)(c)=c$ and $C\subseteq A^G$ which contradicts the fact that $C/A^G=B$ is infinite.

Clearly, for every $b\in B$, we have that $\Gamma(-,b):G\times B\to H$ is continuous and if $s:B\to C$ is continuous at b then for all $g\in G$, we have that Γ is continuous at (g,b). Since by o-minimality, $s:B\to C$ is continuous on a large definable subset of B, by [24, Lemma 2.4], for all $g\in G$ and for all $c\in B$, there is $d\in B$ such that $\Gamma(-,c):G\to H$ is continuous at g and Γ is continuous at $(g,c+d)$. $\square$

Remark 4.3. Let $A,B\leqslant C$ be definable abelian groups. Then $(A+B)/(A\cap B)$ is definably isomorphic to $A/A\cap B\oplus B/A\cap B$ via the map $h:A+B\to A/A\cap B\oplus B/A\cap B$, given by $h(c)=(a+A\cap B)\oplus(b+A\cap B)$ for some $a\in A$ and $b\in B$ such that $c=a+b$. This is a well defined definable homomorphism with kernel $A\cap B$. Moreover, $\dim(A/A\cap B)+\dim(B/A\cap B)=\dim(A/A\cap B\oplus B/A\cap B)=\dim((A+B)/(A\cap B))\leqslant\dim(A+B)$, and $\dim A=\dim(A/A\cap B)+\dim(A\cap B)$.

Lemma 4.4. *Let A be a definable abelian group. Then there is no infinite definable family of definable subgroups of A of dimension zero.*

Proof. Let $\{B(x):x\in X\}$ be a definable family of definable subgroups of A of dimension zero. For each $x\in X$, we see that $B(x)$ is a finite subgroup of A of order, say $n(x)$. By o-minimality, there is an m such that $n(x)\leqslant m$ for all $x\in X$. Let $n=m!$ and let $[n]:A\to A$ be the definable homomorphism which sends a into na. Then $\mathrm{Ker}[n]$ is a definable subgroup of A of bounded exponent and $\cup\{B(x):x\in X\}\subseteq\mathrm{Ker}[n]$. By [27, Corollary 5.8], $E=\mathrm{Ker}[n]$ is finite. $\square$

Lemma 4.5. *Let $\Gamma:G\times B\to H$ be as in* Lemma 4.2. *Then there is a definable group B' of the form B/E where E is a definable subgroup of B of dimension zero and a definable family $\Phi:G\times B'\to B'$ of definable endomorphisms of B' such that* $\mathrm{Ker}_G\,\Phi$ *is a proper definable subgroup of G and for each $g\in G\setminus\mathrm{Ker}_G\Phi$, the map $\Phi(g)$ is a definable automorphism of B'. Moreover, for all $g\in G$ and $c\in B'$, there is a $d\in B'$ such that $\Phi(-,c):G\to B'$ is continuous at g and $\Phi:G\times B'\to B'$ is continuous at $(g,c+d)$.*

Proof. Note that B is definably-connected. Since B has no infinite proper definable subgroups, for each $g \in G$, it follows that $\Gamma(g)(B)$ is either 0 or infinite (with the same dimension as B), definably-connected and with no infinite proper definable additive subgroups. So for all $g, h \in G$, either $\Gamma(g)(B) \cap \Gamma(h)(B)$ has dimension zero or $\Gamma(g)(B) = \Gamma(h)(B)$. 4.3, there is a minimal $n \geqslant 1$ such that for each $i \in \{1, \ldots, n\}$ there is a $g_i \in G$ such that: (i) $\Gamma(g_i)(B) \neq 0$; (ii) $\Gamma(G)(B) \subseteq \Gamma(g_1)(B) + \cdots + \Gamma(g_n)(B)$ and (iii) $F = \bigcap_{i=1}^{n} \Gamma(g_i)(B)$ has dimension zero. The group $D = \Gamma(G)(B)/F$ is definable and we have a natural induced definable family $\Lambda : G \times B \to D$ of definable homomorphisms from B into D. It is easy to see that $\mathrm{Ker}_G \Lambda \neq G$. Now for each $i \in \{1, \ldots, n\}$ let $D_i = \Gamma(g_i)(B)/F$. Then by 4.3, $D = \bigoplus_{i=1}^{n} D_i$ and we have natural induced definable families $\Lambda_i : G \times B \to D_i$ of definable homomorphisms from B into D_i, and there is $i_0 \in \{1, \ldots, n\}$ such that $\mathrm{Ker}_G \Lambda_{i_0} \neq G$.

For each $g \in G \setminus \mathrm{Ker}_G \Lambda_{i_0}$, $\mathrm{Ker}_G \Lambda_{i_0}(g)$ is a definable subgroup of B of dimension zero. So by Lemma 4.4, there is a definable subgroup E of B of dimension zero such that $\cup\{\mathrm{Ker}_G \Lambda_{i_0}(g) : g \in G \setminus \mathrm{Ker}_G \Lambda_{i_0}\} \subseteq E$. Let $B' = B/E$. It is easy to see that Λ_{i_0} induces a natural definable family $\Phi' : G \times B' \to D_{i_0}$ of definable homomorphisms of B' into D_{i_0} such that $\mathrm{Ker}_G \Phi' \neq G$ and for each $g \in G \setminus \mathrm{Ker}_G \Phi'$, $\Phi'(g)$ is a definable injective homomorphism of B' into D_{i_0}. Let $g \in G \setminus \mathrm{Ker}_G \Phi'$. Then $\Phi'(g)(B')$ is a definable subgroup of D_{i_0} of the same dimension as D_{i_0}. Since D_{i_0} is definably-connected, $\Phi'(g)(B') = D_{i_0}$. Therefore, Φ' induces a natural definable family $\Phi : G \times B' \to B'$ of definable endomorphisms of B' such that $\mathrm{Ker}_G \Phi \neq G$ and for each $g \in G \setminus \mathrm{Ker}_G \Phi$, it follows that $\Phi(g)$ is a definable automorphism of B'.

Since for all $g \in G$ and $c \in B$, there is a $d \in B$ such that $\Gamma(-, c) : G \to H$ is continuous at g and $\Gamma : G \times B \to H$ is continuous at $(g, c + d)$, by construction the same holds for Λ and Λ_{i_0}. This also holds for Φ' since it holds for Λ_{i_0} and the definable subset of B' on which a definable section $t : B' \to B$ is continuous is a large definable subset. Finally the result holds for Φ since it holds for Φ'. $\square$

Theorem 4.6. *Let (A, γ) be a definably-connected, non-trivial definable G-module, where G is an infinite definably-connected definable group. Let B' and $\Phi : G \times B' \to B'$ be as in Lemma* 4.5. *Then B' is not definably compact. In particular, A is not definably compact.*

Proof. Let B', B and $\Phi : G \times B' \to B'$ be as in Lemma 4.5. And suppose that B' is definably compact. Since $\mathrm{Ker}_G \Phi \neq G$ and G is definably-connected, we have $\dim(\mathrm{Ker}_G \Phi) < \dim G$ and by Lemma 4.1 there is a definable continuous embedding $\sigma : (a, b) \to G$ such that $\lim_{t \to a^+} \sigma(t) = 1$ and $\sigma(a, b) \subseteq G \setminus \mathrm{Ker}_G \Phi$. Let $x_0 \in B' \setminus \{0\}$. Then for every $t \in (a, b)$, the map $\Phi(\sigma(t), -) : B' \to B'$ is a definable automorphism of B' and therefore, there exists a unique $x \in B'$ such that $\Phi(\sigma(t), x) = x_0$. This gives us a definable function $\tau : (a, b) \to \tau(a, b) \subseteq B'$. Since B' is definably compact, there is an element $c \in B'$ such that $\lim_{t \to a^+} \tau(t) = c$.

By Lemma 4.5, there is $d \in B'$ such that Φ is continuous at $(1, c + d)$ and the definable function $\Phi(-, d) : G \to B'$ is continuous. Then we have $\Phi(\sigma(t), \tau(t) + d) = \Phi(\sigma(t), \tau(t)) + \Phi(\sigma(t), d) = x_0 + \Phi(\sigma(t), d)$ and, taking the limit as $t \to a^+$, we get $0 = \Phi(1, c + d) = x_0 + \Phi(1, d) = x_0$ which is a contradiction.

Suppose that A is definably compact. By Lemma 3.14, A^G, A/A^G, B and B' are definably compact. □

The next corollary was also proved (using the theory of $\bigvee$-definable groups) in [22, Corollary 5.4] but under the assumption that $\mathcal{N}$ has definable Skolem functions. Recall from [17] that a definable group G is *monogenic* if there is $g \in G$ such that the smallest definable subgroup of G containing g (which exists by DCC) is G.

Fact 4.7 (27, Lemma 5.16). *Let $A \trianglelefteq U$ be definable groups. If $A \subseteq Z(U)$ and U/A is monogenic then U is abelian.*

Corollary 4.8. *Let U be a definably compact, definably-connected definable group. Then U is either abelian or $U/Z(U)$ is a definably semi-simple definable group. In particular, if U is solvable then it is abelian.*

Proof. We may assume without loss of generality that $\mathcal{N}$ is $\aleph_0$-saturated. Suppose that $U/Z(U)$ is infinite and not definably semi-simple. Then there is an infinite, abelian, normal, definably-connected, definable subgroup Z of $U/Z(U)$. By Remark 3.12 there is a definable normal subgroup V of U containing $Z(U)$ (and so $Z(U) \subseteq Z(V)$) such that $Z = V/Z(U)$. Therefore V is solvable and $\dim V > 0$. By Lemma 3.14, V is definably compact. If $\dim V < \dim U$, then by induction V^0 is abelian. In this case let $X = V^0$. If $\dim V = \dim U$, then $V = U$ and $U/Z(U)$ is infinite and abelian. Therefore by [27, Corollary 5.8], there is an infinite, definably-connected definably compact monogenic definable subgroup Y of $U/Z(U)$. By Remark 3.12 there is a definable normal subgroup W of U containing $Z(U)$ (and so $Z(U) \subseteq Z(W)$) such that $Y = W/Z(U)$. By Fact 4.7, W is abelian and $\dim W > 0$. In this case, let $X = W^0$.

Now X is a definable U-module by conjugation and by Theorem 4.6, we have $X = X^U \leqslant Z(U)$ which is a contradiction. □

We finish this section with some applications of the results above to $\mathcal{I}$-definable solvable groups globally over I where $\mathcal{I}$ is a definable o-minimal expansion of an ordered group.

Corollary 4.9. *Let A be an $\mathcal{I}$-definable abelian group globally over I. Suppose that (A, γ) is a non-trivial definable G-module where G is an infinite definably-connected definable group. Then there is a definable o-minimal expansion of a real closed field $\mathcal{J}$ which is a definable expansion of $\mathcal{I}$ and there are $\mathcal{J}$-definable subgroups B and C of A such that $B < C$ and C/B is $\mathcal{J}$-definably isomorphic to the additive group of $\mathcal{J}$.*

Proof. Let $\Phi : G \times B' \to B'$ be as in Lemma 4.5. Then by Lemma 3.17, B' is a one-dimensional torsion-free ordered $\mathcal{I}'$-definable group with domain I and identity 0 and $\mathrm{Ker}_G\, \Phi \neq G$, where $\mathcal{I}'$ is the definable o-minimal expansion of $\mathcal{I}$ obtained by adding a predicate for $B = A^G$, B' and C (which is the definable subgroup of A such that $C/B = B'$).

By Lemma 4.1 there is a continuous definable embedding $\sigma:(a,b) \subseteq N \to G$ such that $\lim_{t\to a^+}\sigma(t)=1$ and $\sigma((a,b)) \subseteq G \setminus \mathrm{Ker}_G\,\Phi$. For each $t\in(a,b)$, the map $\Phi(\sigma(t),-):B' \to B'$ is a definable automorphism of B'. Let $z_0\in B'\setminus\{0\}$. Since B' is monogenic, $\Phi(\sigma(t),-)$ is determined by $\Phi(\sigma(t),z_0)$. On the other hand, since $\Phi(-,z_0):G\to B'$ is continuous, $\Phi(\sigma((a,b)),z_0)=(0,e)\subseteq B'$. Therefore, there is an infinite definable family $\Psi:(0,e)\times B'\to B'$ of definable automorphisms of B' given by $\Psi(x,b)=\Phi(y,b)$ for some (equivalently for all) $y\in\sigma((a,b))\subseteq G$ such that $x=\Phi(y,z_0)$. If $\mathscr{J}$ is the definable o-minimal expansion of $\mathscr{I}'$ obtained by adding a predicate for Ψ, then by [14, Lemma 1.7], $\mathscr{J}$ is not linearly bounded with respect to the group operation of B' and so there is an $\mathscr{J}$-definable real closed field whose additive group is B'. □

Corollary 4.10. *Let U be an $\mathscr{I}$-definable solvable group globally over I which is not abelian. Then there is a definable o-minimal expansion of a real closed field $\mathscr{J}$ which is a definable expansion of $\mathscr{I}$ and there are $\mathscr{J}$-definable subgroups B and C of U such that $B<C$ and C/B is $\mathscr{J}$-definably isomorphic to the additive group of $\mathscr{J}$.*

Proof. We may assume without loss of generality that $\mathscr{N}$ is $\aleph_0$-saturated. Since $U/Z(U)$ is infinite and solvable, there is an infinite, abelian, normal, definably-connected, definable subgroup Z of $U/Z(U)$. By Remark 3.12 there is a definable normal subgroup V of U containing $Z(U)$ (and so $Z(U)\subseteq Z(V)$) such that $Z=V/Z(U)$. Therefore V is solvable and $\dim V>0$. By Lemma 3.17, V is an $\mathscr{I}$-definable solvable group globally over I. If V is abelian, put $X=V$. Suppose that $\dim V<\dim U$ and V is not abelian. Then by induction, the result holds. If $\dim V=\dim U$ and V is not abelian, then $V=U$ and $U/Z(U)$ is infinite and abelian. By [27, Corollary 5.8], there is an infinite, definably-connected, monogenic definable subgroup Y of $U/Z(U)$. By Remark 3.12 there is a definable normal subgroup W of V containing $Z(U)$ (and so $Z(U)\subseteq Z(W)$) such that $Y=W/Z(U)$. By Fact 4.7, W is abelian and $\dim W>0$. In this case put $X=W$. In both cases, X is a non-trivial definable U-module under conjugation since $X^U\leqslant Z(U)$. Now the result follows from Corollary 4.9. □

5. Definable solvable groups

5.1. Preliminary lemmas

In this subsection, $\mathscr{I}$ will be a maximal definable o-minimal expansion of an ordered group $(I,0,+,<)$. Recall that if G is a one-dimensional, torsion-free, definably-connected, definable group, then G is an abelian, divisible, ordered definable group with no non-trivial proper definable subgroups. In this case, if A is a definable group and $f:G\to A$ a definable continuous map, by $\lim_{x\to+\infty}f(x)\in A$, we mean that this limit exists and is an element of A. Moreover, we will say that G is globally orthogonal to $\mathscr{I}$ if the definable o-minimal structure induced by $\mathscr{N}$ on the ordered definable group G is globally orthogonal to $\mathscr{I}$.

Lemma 5.1. *Let A be a definably compact definable group. Suppose that G is a one-dimensional, torsion-free, definably-connected, definable group, $(A,\theta)\in EK_{\mathcal{N}}(G,B)$ and $(U,j)\in \mathrm{Ext}_{\mathcal{N}}(G,A,\theta)$. Then U is definably isomorphic to $A\times G$.*

Proof. Let $s:G\to U$ and let $\alpha_{U,s}$ and $h_{\alpha_{U,s}}$ be as in Proposition 3.23. Since A is definably compact, for all $x\in G$, the limit $\lim_{y\to+\infty} h_{\alpha_{U,s}}(x,y)$ exists in A.

For each $x\in G$, let $g_{\alpha_{U,s}}(x)=\lim_{y\to+\infty} h_{\alpha_{U,s}}(x,y)\in A$. By Eq. (2) we have $h_{\alpha_{U,s}}(x,y)=\alpha_{U,s}(x)(h_{\alpha_{U,s}}(y,z))h_{\alpha_{U,s}}(x,yz)(h_{\alpha_{U,s}}(xy,z))^{-1}$. Taking the limit as $z\to+\infty$ (note that, since G is an ordered group $yz\to+\infty$ as $z\to+\infty$) we obtain for all $x,y\in G$, $h_{\alpha_{U,s}}(x,y)=\alpha_{U,s}(x)(g_{\alpha_{U,s}}(y))g_{\alpha_{U,s}}(x)(g_{\alpha_{U,s}}(xy))^{-1}$ which is Eq. (5). By Proposition 3.24, this implies that U is definably isomorphic to $A\rtimes_{\gamma} G$ for some homomorphism $\gamma:G\to \mathrm{Aut}_{\mathcal{N}}(A)$ such that the induced map $\gamma:G\times A\to A$ is definable.

By Theorem 4.6, (A,γ) is a trivial definable G-module, and so U is definably isomorphic to $A\times G$. $\square$

Lemma 5.2. *Let $A=(I,0,+)$ and $G=(I,0,\oplus)$ be $\mathcal{I}$-definably-connected one-dimensional torsion-free $\mathcal{I}$-definable groups. Suppose that (A,γ) is an $\mathcal{I}$-definable G-module and $\mathcal{I}$ is linearly bounded with respect to $+$. If $(U,j)\in\mathrm{Ext}_{\mathcal{N}}(G,A,\gamma)$, then U is definably isomorphic to $A\times G$.*

Proof. By Corollary 4.9, (A,γ) is a trivial definable G-module.

Let $c\in Z^2_{\mathcal{I}}(G,A,\gamma)$ be as in Proposition 3.27. Since $\mathcal{I}$ is linearly bounded with respect to $+$, by [14, Proposition 3.2], there are $r_1,\dots,r_l\in\Lambda(\mathcal{I})$ such that for each $x,y\in G$ we have $c(x,y)=r_x y+o(x,y)$ where $r_x\in\{r_1,\dots,r_l\}$ and $o:G\times G\to A$ is a definable function such that, for each $x\in G$, the definable function $o_x:G\to A$, define by $o_x(y)=o(x,y)$ is bounded (in particular, $\lim_{y\to+\infty} o(x,y)\in A$).

Let $g,h,k\in G$, and suppose h is large enough so that $r_h=r_{g\oplus h}=r$. Then by Eq. (6) we have

$$
\begin{aligned}
&c(h,k)-c(g\oplus h,k)+c(g,h\oplus k)-c(g,h)\\
&\quad=[r_g(h\oplus k)+o(g,h\oplus k)]-[r_g(h)+o(g,h)]+[o(h,k)-o(g\oplus h,k)]\\
&\quad=0.
\end{aligned}
$$

Therefore for all $g\in G$, we see that $r_g=0$, since the above equality implies that r_g is bounded (take $k\to+\infty$). And so, for all $g\in G$, we find that $\lim_{h\to+\infty} c(g,h)\in A$.

For each $g\in G$ let $b(g)=\lim_{k\to+\infty} c(g,k)\in A$. For all $g,h,k\in G$ we have $c(h,k)-c(g\oplus h,k)+c(g,h\oplus k)-c(g,h)=0$ by equation (6). Taking the limit as $k\to+\infty$ (note that, since G is an ordered group $hk\to+\infty$ as $k\to+\infty$) we obtain $c(g,h)=b(h)-b(g\oplus h)+b(g)$. Therefore, by Proposition 3.28, U is definably isomorphic to $A\times G$. $\square$

Lemma 5.3. *Let G be a one-dimensional, definably-connected, torsion-free definable group and let A be an $\mathcal{I}$-definable solvable group globally over I. Suppose that G and $\mathcal{I}$ are globally orthogonal, $(A,\theta)\in EK_{\mathcal{N}}(G,B)$ and $(U,j)\in\mathrm{Ext}_{\mathcal{N}}(G,A,\theta)$. Then U is definably isomorphic to $A\times G$.*

Proof. Let $s: G \to U$ and let $\alpha_{U,s}$ and $h_{\alpha_{U,s}}$ be as in Proposition 3.23. Let $x \in G$. If $\lim_{y\to+\infty} h_{\alpha_{U,s}}(x,y)$ does not exist in A, then by the monotonicity theorem, for some $q \in \{1,\dots,n\}$, the definable map $b^q: G \to I$ given by $b^q(y) = \pi^q(h_{\alpha_{U,s}}(x,y))$ where $\pi^q: I^n \to I$ is the projection onto the q-coordinate, determines a definable bijection between an unbounded subinterval K in G and the unbounded subinterval $b^q(K)$ of I. But since we have definable group structures on I and on G, this definable bijection can be extended to a definable bijection between I and G, which is a contradiction. Therefore, for all $x \in G$, $\lim_{y\to+\infty} h_{\alpha_{U,s}}(x,y) \in A$.

For each $x \in G$, let $g_{\alpha_{U,s}}(x) = \lim_{y\to+\infty} h_{\alpha_{U,s}}(x,y) \in A$. By Eq. (2) we have $h_{\alpha_{U,s}}(x,y) = \alpha_{U,s}(x)(h_{\alpha_{U,s}}(y,z))h_{\alpha_{U,s}}(x,yz)(h_{\alpha_{U,s}}(xy,z))^{-1}$. Taking the limit as $z \to +\infty$ (note that, since G is an ordered group $yz \to +\infty$ as $z \to +\infty$) we obtain for all $x,y \in G$, $h_{\alpha_{U,s}}(x,y) = \alpha_{U,s}(x)(g_{\alpha_{U,s}}(y))g_{\alpha_{U,s}}(x)(g_{\alpha_{U,s}}(xy))^{-1}$ which is equation (5). By Proposition 3.24, this implies that U is definably isomorphic to $A \rtimes_\gamma G$ for some homomorphism $\gamma: G \to \mathrm{Aut}_{\mathcal{N}}(A)$ such that the induced map $\gamma: G \times A \to A$ is definable.

To finish, we need to show that $\gamma(g)(a) = a$ for all $a \in A$ and $g \in G$. Suppose that this is not the case.

Suppose that A is abelian. Then (A,γ) is a definable G-module. Let B' and $\Phi: G \times B' \to B'$ be as in Lemma 4.5. Then B' is a one-dimensional torsion-free ordered $\mathcal{I}$-definable group with domain I and $\mathrm{Ker}_G\Phi = 1$. Let $x_0 \in B' \setminus \{0\}$. For each $t \in G$, $\Phi(t,-): B' \to B'$ is a definable automorphism of B'. Therefore there is a definable map $\tau: G \to B'$ such that $\Phi(t,\tau(t)) = x_0$ for all $t \in G$. Since $\mathcal{I}$ and G are globally orthogonal, there is $c \in B'$ such that $\lim_{t\to-\infty}\tau(t) = c$. By Lemma 4.5, there is $d \in B'$ such that Φ is continuous at $(1, c+d)$ and the definable function $\Phi(-,d): G \to B'$ is continuous. Then we have $\Phi(t,\tau(t)+d) = \Phi(t,\tau(t)) + \Phi(t,d) = x_0 + \Phi(t,d)$ and, taking the limit as $t \to -\infty$, we get $0 = \Phi(1,c+d) = x_0 + \Phi(1,d) = x_0$ which is a contradiction.

Suppose on the other hand that A is not abelian. Then by Corollary 4.10, $\mathcal{I}$ is a definable o-minimal expansion of a real closed field. By [19, Corollary 2.22 and Fact 2..25] we have, after fixing a basis for the tangent space of each A, a definable homomorphism $\alpha: G \to GL(m,I)$ defined by $\alpha(g) = d_0(\gamma(g))$ and with kernel $\{g \in G : \gamma(g)(a) = a \text{ for all } a \in A\} = 1$. So G is in definable bijection with a one-dimensional, definably-connected definable subset $\alpha(G)$ of I^{m^2}. But since $\mathcal{I}$ expands a real closed field, there is an $\mathcal{I}$-definable bijection between $\alpha(G)$ and I. And so there is a definable bijection between G and I which is again a contradiction. □

Let U be a definable abelian group of dimension two and with no definably compact parts. Lemmas 5.2 and 5.3 above show that either U is definably isomorphic to a direct product of two one-dimensional torsion-free definable groups, or U is a definable group in a definable o-minimal expansion $\mathcal{I}$ of a real closed field $(I,0,1,+,\cdot,<)$ and there is an $\mathcal{I}$-definable extension $1 \to A \to U \to G \to 1$ where $A = (I,0,+)$ and $G = (I,0,\oplus)$ is a one-dimensional torsion-free $\mathcal{I}$-definable group.

Before we consider the latter case, we prove the following lemma which is related to the Miller–Starchenko problem we mentioned in the introduction.

Lemma 5.4. *Let $\mathcal{I} = (I,0,1,+,\cdot,<,\dots)$ be a definable o-minimal expansion of a real closed field and let $G = (I,0,\oplus)$ be an $\mathcal{I}$-definable one-dimensional torsion-free* *ordered group. Then G is $\mathscr{I}$-definably isomorphic to $(I,0,+)$ if and only if there is an $\mathscr{I}$-definable C^1 function $\alpha: G \to I$ such that $\alpha(0)=0$, $\alpha'(0) \neq 0$ and $\alpha'(t)\frac{\partial \oplus}{\partial x}(0,t) = \alpha'(0)$ for all $t \in G$, where $\oplus(x,t) = x \oplus t$ for all $x \in G$.*

Proof. For $x,t \in G$, let $\lambda_t(x) = x \oplus t$. Then for all $s,t \in G$, we have $\frac{\mathrm{d}\lambda_t}{\mathrm{d}x}(s) = \frac{\partial \oplus}{\partial x}(s,t)$ where $\oplus(s,t) = s \oplus t$.

Suppose that $\alpha: G \to (I,0,+)$ is an $\mathscr{I}$-definable isomorphism. Then α is C^1 with $\alpha(0)=0$, and for all $x,t \in G$ we have $\alpha(\lambda_t(x)) = \alpha(x \oplus t) = \alpha(x) + \alpha(t)$. Taking the derivative with respect to x in this equation, we get $\alpha'(\lambda_t(x))\frac{\mathrm{d}\lambda_t}{\mathrm{d}x}(x) = \alpha'(x)$. Putting $x=0$ we get $\alpha'(t)\frac{\mathrm{d}\lambda_t}{\mathrm{d}x}(0) = \alpha'(0)$. By associativity of $\oplus$, for all $t,s \in G$, we have $\frac{\mathrm{d}\lambda_{s\oplus t}}{\mathrm{d}x}(0) = \frac{\mathrm{d}\lambda_t}{\mathrm{d}x}(s)\frac{\mathrm{d}\lambda_s}{\mathrm{d}x}(0)$. Therefore, $\frac{\mathrm{d}\lambda_t}{\mathrm{d}x}(0) \neq 0$ and so $\alpha'(0) \neq 0$.

Let $\alpha: G \to I$ be an $\mathscr{I}$-definable C^1 function such that $\alpha(0)=0$, $\alpha'(0) \neq 0$ and for all $t \in G$, $\alpha'(t)\frac{\mathrm{d}\lambda_t}{\mathrm{d}x}(0) = \alpha'(0)$. Replace in this equation t by $s \oplus t$, then we get $\alpha'(\lambda_t(s))\frac{\mathrm{d}\lambda_{s\oplus t}}{\mathrm{d}x}(0) = \alpha'(0)$. Using the equation obtained above from the associativity of $\oplus$, we get $\alpha'(\lambda_t(s))\frac{\mathrm{d}\lambda_t}{\mathrm{d}x}(s)\frac{\mathrm{d}\lambda_s}{\mathrm{d}x}(0) = \alpha'(0)$. But $\alpha'(s)\frac{\mathrm{d}\lambda_s}{\mathrm{d}x}(0) = \alpha'(0)$ and therefore, after dividing both sides of this equation by $\frac{\mathrm{d}\lambda_s}{\mathrm{d}x}(0)$, we get $\alpha'(\lambda_t(s))\frac{\mathrm{d}\lambda_t}{\mathrm{d}x}(s) = \alpha'(s)$. This implies that for each $t \in G$, the definable function $\beta: G \to I$ given by $\beta(x) = \alpha(\lambda_t(x)) - \alpha(x) - \alpha(t)$ is such that $\frac{\mathrm{d}\beta}{\mathrm{d}x}(s) = 0$ for all $s \in G$, i.e. α is an $\mathscr{I}$-definable isomorphism. $\square$

Lemma 5.5. *Suppose that $\mathscr{I}$ is an expansion of a real closed field and that we have an $\mathscr{I}$-definable abelian extension $1 \to A \to U \to G \to 1$ where $A = (I,0,+,<)$ and $G = (I,0,\oplus,<)$ is a one-dimensional torsion-free $\mathscr{I}$-definable group. Let $m \in \mathbb{N}$. Then there is a 2-cocycle $c \in Z^2_{\mathscr{I}}(G,A)$ corresponding to this $\mathscr{I}$-definable extension and there is $\varepsilon > 0$ such that c is C^m everywhere except possibly on $\{\ominus\varepsilon\} \times G \cup G \times \{\ominus\varepsilon\}$. Moreover, U is $\mathscr{I}$-definably isomorphic to $A \times G$ iff there is an $\mathscr{I}$-definable function $\alpha: G^{>\ominus\varepsilon} \to A$ such that*

$$\forall s \in G^{>\ominus\varepsilon},\ \alpha'(s)\frac{\partial \oplus}{\partial x}(0,s) = \alpha'(0) - \frac{\partial c}{\partial x}(0,s).$$

Proof. Let $t: G \to U$ be an $\mathscr{I}$-definable section. Then by o-minimality there are $g_0 > \varepsilon > 0$ such that t is C^m on $(g_0 \ominus \varepsilon, +\infty)$. Let $s: G \to U$ be the $\mathscr{I}$-definable section given by: for all $g \in G$, if $g > \ominus\varepsilon$ then $s(g) = t(g \oplus g_0)t(g_0)^{-1}$ and if $g \leqslant \ominus\varepsilon$ then $s(g) = s(\ominus g)^{-1}$. Then $s(0)=0$ and s is C^m on $G \setminus \{\ominus\varepsilon\}$. Let $c(g,h) = s(g)s(h)s(g \oplus h)^{-1}$ be the corresponding $\mathscr{I}$-definable 2-cocycle. Then c is C^m everywhere except possibly on $\{\ominus\varepsilon\} \times G \cup G \times \{\ominus\varepsilon\}$.

By Proposition 3.27 U is $\mathscr{I}$-definably isomorphic to an $\mathscr{I}$-definable group V with domain $A \times G$ and group operation given by $(a,x)(b,y) = (a+b+c(x,y),xy)$. By Proposition 3.28, V (and therefore U) is $\mathscr{I}$-definably isomorphic with $A \times G$ if and only if there is an $\mathscr{I}$-definable function $\alpha: G \to A$ with $\alpha(0)=0$ such that the definable function $\beta: G \to U$, $\beta(s) = (-\alpha(s),s)$ is a definable homomorphism. Or equivalently, if and only if there is an $\mathscr{I}$-definable function $\alpha: G^{>\ominus\varepsilon} \to A$ with $\alpha(0)=0$ such that the definable function $\beta: G^{>\ominus\varepsilon} \to U$, $\beta(s) = (-\alpha(s),s)$ is a definable partial homomorphism (because such an $\mathscr{I}$-definable partial homomorphism $\beta: G^{>\ominus\varepsilon} \to U$ can easily be extended to an $\mathscr{I}$-definable homomorphism $\gamma: G \to U$ and so V is

$\mathscr{I}$-definably isomorphic to $A \times G$). Equivalently, if and only if there is an $\mathscr{I}$-definable function $\alpha : G^{>\ominus\varepsilon} \to A$ such that

$$\forall t, x \in G^{>\ominus\varepsilon}, \alpha(\lambda_t(x)) = \alpha(x) + \alpha(t) - c(x,t), \tag{a}$$

or equivalently

$$\forall t, x \in G^{>\ominus\varepsilon}, \alpha'(\lambda_t(x)) \frac{\mathrm{d}\lambda_t}{\mathrm{d}x}(x) = \alpha'(x) - \frac{\partial c}{\partial x}(x,t). \tag{b}$$

Putting $x = 0$ and $t = s$ in equation (b) we get

$$\forall s \in G^{>\ominus\varepsilon},\ \alpha'(s) \frac{\mathrm{d}\lambda_s}{\mathrm{d}x}(0) = \alpha'(0) - \frac{\partial c}{\partial x}(0,s). \tag{c}$$

To prove the converse replace s by $s \oplus t$ in Eq. (c). Then we get $\alpha'(\lambda_t(s)) \frac{\mathrm{d}\lambda_{s\oplus t}}{\mathrm{d}x}(0) = \alpha'(0) - \dfrac{\partial c}{\partial x}(0, s \oplus t)$. By associativity of $\oplus$, we get $\frac{\mathrm{d}\lambda_{s\oplus t}}{\mathrm{d}x}(0) = \frac{\mathrm{d}\lambda_t}{\mathrm{d}x}(s)\frac{\mathrm{d}\lambda_s}{\mathrm{d}x}(0)$. Therefore, $\alpha'(\lambda_t(s))\frac{\mathrm{d}\lambda_t}{\mathrm{d}x}(s)\frac{\mathrm{d}\lambda_s}{\mathrm{d}x}(0) = \alpha'(0) - \frac{\partial c}{\partial x}(0, s \oplus t)$. On the other hand, if in Eq. (6) we put $g = x$, $h = s$, $k = t$ and take the derivative with respect to x and put $x = 0$, we get $-\frac{\partial c}{\partial x}(s,t)\frac{\mathrm{d}\lambda_s}{\mathrm{d}x}(0) + \frac{\partial c}{\partial x}(0, s \oplus t) - \frac{\partial c}{\partial x}(0,s) = 0$. Using this we get $\alpha'(\lambda_t(s))\frac{\mathrm{d}\lambda_t}{\mathrm{d}x}(s)\frac{\mathrm{d}\lambda_s}{\mathrm{d}x}(0) = \alpha'(0) - \frac{\partial c}{\partial x}(0,s) - \frac{\partial c}{\partial x}(s,t)\frac{\mathrm{d}\lambda_s}{\mathrm{d}x}(0)$. But $\alpha'(s)\frac{\mathrm{d}\lambda_s}{\mathrm{d}x}(0) = \alpha'(0) - \frac{\partial c}{\partial x}(0,s)$ by Eq. (c). Thus $\alpha'(\lambda_t(s))\frac{\mathrm{d}\lambda_t}{\mathrm{d}x}(s)\frac{\mathrm{d}\lambda_s}{\mathrm{d}x}(0) = \alpha'(s)\frac{\mathrm{d}\lambda_s}{\mathrm{d}x}(0) - \frac{\partial c}{\partial x}(s,t)\frac{\mathrm{d}\lambda_s}{\mathrm{d}x}(0)$. And, after dividing both sides of this equation by $\frac{\mathrm{d}\lambda_s}{\mathrm{d}x}(0)$, we get Eq. (b). □

5.2. The main theorems

Remark 5.6. Suppose that A is a definably-connected definable solvable group of the form $A = K \times A_1 \times \cdots \times A_r$ where K is definably compact and definably-connected and where, for each $i = 1, \ldots, r$, there is a definable o-minimal expansion $\mathscr{I}_i$ of an ordered group pairwise globally orthogonal such that A_i is an $\mathscr{I}_i$-definable group globally over I_i. Then $\mathrm{Aut}_{\mathscr{N}}(A) = \mathrm{Aut}_{\mathscr{N}}(K) \times \mathrm{Aut}_{\mathscr{N}}(A_1) \times \cdots \times \mathrm{Aut}_{\mathscr{N}}(A_r)$.

In fact, let $\alpha : A \to A$ be a definable automorphism of A. Then, since K is the maximal definably compact, definably-connected definable subgroup of A, by Lemmas 3.14 and 3.15, $\alpha(K)$ is a definably compact, definably-connected definable subgroup of A, and so $\alpha(K) \subseteq K$. For $i = 1, \ldots, k$, the subgroup A_i is the maximal $\mathscr{I}_i$-definable subgroup of A globally over I_i. By Lemma 3.17, $\alpha(A_i)$ is an $\mathscr{I}_i$-definable subgroup of A globally over I_i. Therefore, $\alpha(A_i) \subseteq A_i$.

The following important result is proved by Peterzil and Steinhorn.

Theorem 5.7 (Theorem 1.2 [23]). *Let G be a definable group which is not definably compact. Then G has a one-dimensional torsion-free ordered definable subgroup.*

We are ready to prove one of our main results describing definable solvable groups.

Theorem 5.8. *Suppose that U is a definably-connected definable solvable group. Then U has a definable normal subgroup V such that U/V is a definably compact definable solvable group and* $V = K \times W_1 \times \cdots \times W_s \times V_1 \times \cdots \times V_k$. *Here K is the* *definably-connected, definably compact normal subgroup of U of maximal dimension. For each $j \in \{1,\ldots,s\}$ (resp., $i \in \{1,\ldots,k\}$), there is a semi-bounded o-minimal expansion $\mathcal{J}_j$ of a group (resp., an o-minimal expansion $\mathcal{I}_i$ of a real closed field) definable in $\mathcal{N}$ all of which are pairwise globally orthogonal such that W_j is a direct product of copies of the additive group of $\mathcal{J}_j$ and V_i is definably isomorphic to an $\mathcal{I}_i$-definable solvable group with no $\mathcal{I}_i$-definably compact parts.*

Proof. We prove this by induction on the dimension of U. The result is clearly true for dimension zero. So let U be as above and suppose that the result is true for solvable definable groups of lower dimensions.

We first show the existence of K. If U has no non-trivial definably compact definably-connected definable normal subgroups, then we put $K = 1$. Otherwise, let K' be a definably compact definably-connected definable normal subgroup of U of positive dimension and let $U_1 = U/K'$. Then since $\dim U_1 < \dim U$, it follows that U_1 has a definably-connected definably compact normal, definable subgroup K_1 of maximal dimension. Now apply Remark 3.12 and let K be the definable normal subgroup of U which is a definable extension of K_1 by K'. By Lemmas 3.14 and 3.15, K is a definably compact definably-connected definable normal subgroup of U. We show that K is the unique such definable subgroup of maximal dimension. Let H be a definably compact, definably-connected, definable normal subgroup of U. Since KH/K is definably isomorphic to $H/K \cap H$, by Lemmas 3.14 and 3.15, KH and KH/K are definably compact definably-connected definable groups with KH normal in U and KH/K normal in U_1. Therefore, $KH/K \subseteq K_1$ and $KH \subseteq K$.

Set $U' = U/K$. Then U' is definably-connected (by Lemma 3.15) and has no non-trivial definably compact, definably-connected, definable normal subgroups. In particular, U' is not definably compact. Therefore, by Theorem 5.7 (i.e., [23, Theorem 1.2]), U' has a definably-connected, one-dimensional, torsion-free definable subgroup H. In particular, H has no definably compact parts. For each $u \in U'$, uHu^{-1} is also a one dimensional definable solvable group with no definably compact parts. Moreover, $HuHu^{-1}/H$ is definably isomorphic to $H/H \cap uHu^{-1}$ and, by Lemma 3.18, $H \cap uHu^{-1}$ is also a definable solvable group with no definably compact parts. By Lemma 3.18 again, $HuHu^{-1}$ is a definable solvable group with no definably compact parts $\dim(HuHu^{-1}) = \dim H + \dim(H/H \cap uHu^{-1}) \geqslant \dim H$. Therefore, there is a definable solvable subgroup W of U' with no definably compact parts and of maximal dimension. By exactly the same argument as above, we see that W is a normal subgroup of U' and U'/W is definably compact. Now apply Remark 3.12 and let V be the definable normal subgroup of U which is a definable extension of W by K. Then $U/V = U'/W$ is definably compact. Since W is a definable solvable group with no definably compact parts, there is a normal definable subgroup W' of W such that $G = W/W'$ is a one dimensional definable solvable group with no definably compact parts. Apply Remark 3.12 and let V' be the definable normal subgroup of V which is a definable extension of W' by K. Then $\dim V' < \dim V$, and by the induction hypothesis, the result holds for V' i.e., $V' = K \times W_1' \times \cdots \times W_s' \times V_1' \times \cdots \times V_k'$, where K is the definably-connected, definably compact normal subgroup of V' of maximal dimension. For each $j \in \{1,\ldots,s\}$ (resp., $i \in \{1,\ldots,k\}$) there is a semi-bounded o-minimal expansion $\mathcal{J}_j'$ of a group (resp., an o-minimal expansion $\mathscr{I}'_i$ of a real closed field) definable in $\mathscr{N}$ all of which are pairwise globally orthogonal such that W'_j is a direct product of copies of the additive group of $\mathscr{J}'_j$ and V'_i is definably isomorphic to an $\mathscr{I}'_i$-definable solvable group with no $\mathscr{I}'_i$-definably compact parts.

To finish the proof of the theorem, use Remark 5.6, Remark 3.26, and Lemmas 5.1, 5.2 and 5.3 $\square$

Remark 5.9. Let U be a definably-connected definable abelian group and let $V, K, W_1, \ldots, W_s, V_1, \ldots, V_k$ be the subgroups of U given by Theorem 5.8. If $A \in \{V, K, W_1, \ldots, W_s, V_1, \ldots, V_k\}$, then A is invariant under every definable endomorphism of U.

In fact, let $\alpha : U \to U$ be a definable endomorphism of U and let $B = W_1 \times \cdots \times W_s \times V_1 \times \cdots \times V_k$. Then B is the maximal definable subgroup of U with no definably compact parts. By Lemma 3.18, $\alpha(B)$ is a definable subgroup of U with no definably compact parts, and so $\alpha(B) \subseteq B$. Similarly, $\alpha(K) \subseteq K$ and consequently $\alpha(V) \subseteq V$. For $i = 1, \ldots, k$, the subgroup V_i is the maximal $\mathscr{I}_i$-definable subgroup of U globally over I_i. By Lemma 3.17, $\alpha(V_i)$ is an $\mathscr{I}_i$-definable subgroup of U globally over I_i. Therefore, $\alpha(V_i) \subseteq V_i$ and similarly, $\alpha(W_j) \subseteq W_j$ for all $j = 1, \ldots, s$.

The same argument, shows that if U is solvable, then A is invariant under every definable automorphism of U.

Theorem 5.10. *Let $\mathscr{I} = (I, 0, 1, +, \cdot, <, \ldots)$ be an o-minimal expansion of a real closed field and let U be an $\mathscr{I}$-definable solvable group with no $\mathscr{I}$-definably compact parts. Then $U = W \times V$, where W is the maximal definable subgroup of U which is a direct product of copies of the linearly bounded one-dimensional torsion-free $\mathscr{I}$-definable group. The group V is an $\mathscr{I}$-definable group whose centre $Z(V)$ has an $\mathscr{I}$-definable subgroup Z such that $Z(V)/Z$ is a direct product of copies of the linearly bounded one-dimensional torsion-free $\mathscr{I}$-definable group and such that there are $\mathscr{I}$-definable subgroups $1 < Z_1 < \cdots < Z_m = Z$ where for each $l \in \{1, \ldots, m\}$, the group Z_l/Z_{l-1} is the additive group of $\mathscr{I}$, and $V/Z(V)$ $\mathscr{I}$-definably embeds into $GL(n, I)$.*

Proof. We prove the result by induction on the dimension of U. The result is clearly true for dimension one. So let U be as above and suppose that the result is true for $\mathscr{I}$-definable solvable groups with no $\mathscr{I}$-definably compact parts of lower dimensions than that of U.

Since U is an $\mathscr{I}$-definable solvable group with no $\mathscr{I}$-definably compact parts, there is a normal $\mathscr{I}$-definable subgroup U' of U such that $G = U/U'$ is a one-dimensional $\mathscr{I}$-definable solvable group with no $\mathscr{I}$-definably compact parts. Since $\dim U' < \dim U$, by the induction hypothesis, the result holds for U'. In particular, $U' = W' \times V'$ where W' is the maximal $\mathscr{I}$-definable subgroup of U' which is a direct product of copies of the linearly bounded one-dimensional torsion-free $\mathscr{I}$-definable group and V' is an $\mathscr{I}$-definable solvable group with no $\mathscr{I}$-definably compact parts and with no $\mathscr{I}$-definable subgroups $\mathscr{I}$-definably isomorphic to a direct product of copies of the linearly bounded one-dimensional torsion-free $\mathscr{I}$-definable group. Note that under these conditions, W' and V' are $\mathscr{I}$-definable subgroups of U' invariant under all $\mathscr{I}$-definable automorphisms of U'. By Remark 3.26 and Lemma 5.3, U' is $\mathcal{I}$-definably isomorphic to $W' \times V''$ where V'' is an $\mathcal{I}$-definable extension of G by V'. If G is $\mathcal{I}$-definably isomorphic to the linearly bounded one-dimensional torsion-free $\mathcal{I}$-definable group and V'' is $\mathcal{I}$-definably isomorphic to $G \times V'$, let $W = W' \times G$ and $V = V'$; otherwise, let $W = W'$ and $V = V''$. Clearly, W has the properties mentioned in the theorem and V is an $\mathcal{I}$-definable solvable group with no $\mathcal{I}$-definably compact parts and with no $\mathcal{I}$-definable subgroups $\mathcal{I}$-definably isomorphic to a direct product of copies of the linearly bounded one-dimensional torsion-free $\mathcal{I}$-definable group.

The fact that $Z(V)$ is as described is proved in the same way. The fact that $V/Z(V)$ $\mathcal{I}$-definably embeds into some $GL(n,I)$ is proved in [17, Corollary 3.3]. □

Corollary 5.11 below is an adaption of an argument due to Iwasawa (see the proof of [10, Lemma 3.4]).

Corollary 5.11. *Let $\mathcal{I}=(I,0,1,+,\cdot,<,\ldots)$ be an o-minimal expansion of a real closed field with no Peterzil–Steinhorn $\mathcal{I}$-definable groups. Then every $\mathcal{I}$-definable solvable group U with no $\mathcal{I}$-definable compact parts is $\mathcal{I}$-definably isomorphic to a definable group of the form $U' \times G_1 \cdots G_k \cdot G_{k+1} \cdots G_l$ where U' is a direct product of copies of linearly bounded one-dimensional torsion-free $\mathcal{I}$-definable groups. For $i = 1,\ldots,k$, we have $G_i = (I,0,+)$ and for $i = k+1,\ldots,l$, we have $G_i = (I^{>0},1,\cdot)$.*

Proof. By Theorem 5.10, we may assume that $U = U' \times G$ where U' is the maximal $\mathcal{I}$-definable normal subgroup of U which is a product of copies of the linearly bounded one-dimensional torsion-free $\mathcal{I}$-definable group and G is as described there. Furthermore, since there are no Peterzil–Steinhorn $\mathcal{I}$-definable groups, every $\mathcal{I}$-definable abelian group with no $\mathcal{I}$-definably compact parts is a direct product of one-dimensional torsion-free $\mathcal{I}$-definable groups. Therefore by an argument similar to that used in the proof of Theorem 5.8 (substitute "definably compact, definably-connected definable group" by "linearly bounded one-dimensional torsion-free $\mathcal{I}$-definable group"), we can assume that $Z(G)$ is a direct product of copies of additive group of $\mathcal{I}$ and there are $\mathcal{I}$-definable subgroups $1 = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_{n+1} = G$ such that for each $i \in \{1,\ldots,n\}$, H_i is the smallest definable normal subgroup of H_{i+1} such that H_{i+1}/H_i is abelian, H_i/H_{i-1} is a direct product of copies of additive group of $\mathcal{I}$ and H_{n+1}/H_n is a direct product of copies (possibly zero copies) of the linearly bounded one-dimensional torsion-free $\mathcal{I}$-definable group.

Let $\bar{G} = G/Z(G)$. Since $\bar{G}$ $\mathcal{I}$-definably embeds into some $GL(k,I)$, by [18, Theorem 4.1] and the remark above, $\bar{G} = \bar{G}_1 \cdots \bar{G}_{\bar{k}} \cdot \bar{G}_{\bar{k}+1} \cdots \bar{G}_{\bar{l}}$ where for each $i \in \{1,\ldots,\bar{k}\}$, $\bar{G}_i = (I,0,+)$ and for each $i \in \{\bar{k}+1,\ldots,\bar{l}\}$, $\bar{G}_i = (I^{>0},1,\cdot)$. Let N be the $\mathcal{I}$-definable extension of $\bar{G}_1 \cdots \bar{G}_{\bar{k}} \cdot \bar{G}_{\bar{k}+1} \cdots \bar{G}_{\bar{l}-1}$ by $Z(U)$ (and therefore G/N is a one dimensional torsion-free $\mathcal{I}$-definably-connected $\mathcal{I}$-definable group). By induction it is enough to show that G contains an $\mathcal{I}$-definable subgroup H ($\mathcal{I}$-definably isomorphic with G/N) such that $G = NH$ and $H \cap N = 1$.

We prove this by induction on $\bar{l}$. Note that if $\bar{l} = 0$ or $\bar{l} = 1$, then G is abelian (in the second case by Fact 4.7) and so the claim holds by assumption. Assume that the claim is true all $\mathcal{I}$-definable groups with no $\mathcal{I}$-definably compact parts and with lower $\bar{l}$.

Suppose that N contains a proper $\mathscr{I}$-definable normal subgroup N_1 of G. By induction applied to G/N_1 there is an $\mathscr{I}$-definable subgroup G_1 such that $G=NG_1$, $G_1\cap N=N_1$ and $G_1/N_1=G/N$, Again the induction assumption for G_1 and N_1 gives us an $\mathscr{I}$-definable subgroup H such that $G_1=N_1H$ and $H\cap N_1=1$. This H satisfies the claim.

We can therefore assume that N has no proper $\mathscr{I}$-definable subgroup which is normal in G. If N is in the centre of G then by Fact 4.7, G is abelian and by assumption the claim is proved. If N is not in the centre of G then, using the decomposition series $1=K_0\trianglelefteq K_1\trianglelefteq\cdots\trianglelefteq K_{m+1}=N$ of N like the one we obtained for G above, we see that N must be a direct product of k copies of the additive group of $\mathscr{I}$. Therefore N is an $\mathscr{I}$-definable G-module under conjugation and we have a natural $\mathscr{I}$-definable homomorphism $A:G\to GL(k,I)$. And so there is an $\mathscr{I}$-definable embedding $G/N\to GL(k,I)$. We show that that there is $g\in G$ such that $\det(A(g)-\mathrm{Id})\neq 0$ and so $[N,g]=N$. Since N is not in the centre of G, there is $g\in G$ which does not commute with some element in N. Let N' be the eigenspace for the value 1 of the matrix $A(g)$. Since $A(G)$ is abelian, N' is invariant under all the $A(h)$. But this means that the $\mathscr{I}$-definable subgroup N' of N is normal in G and therefore by the assumption we must have either $N'=N$ or $N'=1$. The first case does not hold since g does not commute with some element of N. Therefore $N'=1$, $\det(A(g)-\mathrm{Id})\neq 0$ and $[N,g]=N$.

Now take an arbitrary element $y\in G$ and put $z=gyg^{-1}y^{-1}$. Since G/N is abelian, we have $z\in N$. Take $u\in N$ such that $z=gug^{-1}u^{-1}$ and put $v=u^{-1}y$. It follows that $gv=vg$ and so $G=NC_G(g)$. If $x\in C_G(g)\cap N$, then $gxg^{-1}x^{-1}=1$ and $\det(A(g)-\mathrm{Id})\neq 0$ implies that $x=1$, i.e. $C_G(g)\cap N=1$. □

Corollary 5.12. *Let $\mathscr{I}$ and U be as in Corollary* 5.11. *Then $G=G_1\cdots G_k\cdot G_{k+1}\cdots G_l$, there is an $\mathscr{I}$-definable embedding $\mathscr{I}$-definably of G into some $GL(n,I)$ and U is $\mathscr{I}$-definably isomorphic to a group definable in one of the reducts $(I,0,1,+,\cdot,\oplus)$, $(I,0,1,+,\cdot,\oplus,e^t)$ or $(I,0,1,+,\cdot,\oplus,t^{b_1},\dots,t^{b_r})$ of $\mathscr{I}$ where $(I,0,\oplus)$ is the Miller–Starchenko group of $\mathscr{I}$, e^t is the $\mathscr{I}$-definable exponential map* (*if it exists*), *and the t^{b_j}'s are $\mathscr{I}$-definable power functions. Moreover*, *if U is nilpotent then U is $\mathscr{I}$-definably isomorphic to a group definable in the reduct $(I,0,1,+,\cdot,\oplus)$ of $\mathscr{I}$.*

Proof. Since $G=G_1\cdots G_k\cdot G_{k+1}\cdots G_l$, an induction on l shows that G $\mathscr{I}$-definably embeds into some $GL(n,I)$ and G is $\mathscr{I}$-definably isomorphic to a group definable in one of the following reducts $(I,0,1,+,\cdot)$, $(I,0,1,+,\cdot,e^t)$ or $(I,0,1,+,\cdot,t^{b_1},\dots,t^{b_r})$ of $\mathscr{I}$ where e^t is the $\mathscr{I}$-definable exponential map (if it exists), and the t^{b_j}'s are $\mathscr{I}$-definable power functions. If U is nilpotent then G is nilpotent and by [19, Proposition 3.10], G is $\mathscr{I}$-definably isomorphic to a group definable in the reduct $(I,0,1,+,\cdot)$ of $\mathscr{I}$. □

Remark 5.13 (Peterzil et al. [19]). There are solvable linear groups U and V definable in o-minimal expansions of $(\mathbb{R},0,1,+,\cdot,<)$ by exp and t^r respectively, such that U (resp., V) is not isomorphic, even abstractly, to a definable group in o-minimal expansions of $(\mathbb{R},0,1,+,\cdot,<)$ by some t^s (resp., to a real semi-algebraic group). For example, take $A=(\mathbb{R}^2,0,+)$, $G=(\mathbb{R},0,+)$ and $H=(\mathbb{R}^{>0},1,\cdot)$. Let $U=A\rtimes_\alpha G$ and $V=A\rtimes_\beta H$, where $\alpha(t)(a,b)=(\exp(t)a+t\exp(t)b,\,\exp(t)b)$ and $\beta(t)(a,b)=(ta,t^rb)$.

We end this subsection with the following result from [23] which shows that definable abelian groups are not necessarily the direct product of a definable abelian group with no definably compact parts and a definably compact definable abelian group.

Fact 5.14 (Peterzil and Steinhorn [23]). Let $\tilde{\mathbb{R}}=(\mathbb{R},0,1,+,<)$. Then for $m,n\in\mathbb{N}$ and L an integral lattice in $\mathbb{R}^n$ there are $\tilde{\mathbb{R}}$-definable abelian groups $T(m,n,L)$ and $T(n,L)$ with dimensions $m+n$ and n respectively, such that we have an $\tilde{\mathbb{R}}$-definable extension $1\to(\mathbb{R}^m,0,+)\to T(m,n,L)\to T(n,L)\to 1$. Moreover, if L is generic then $(\mathbb{R}^m,0,+)$ does not have an $\tilde{\mathbb{R}}$-definable complement in $T(m,n,L)$ and $T(n,L)$ does not have $\tilde{\mathbb{R}}$-definable infinite proper subgroups.

The same result holds in $(\mathbb{R},0,1,+,\cdot,<)$.

6. The Lie–Kolchin–Mal'cev theorem

6.1. More on definable G-modules

In this subsection we will describe definable G-modules, generalising a result from [12] describing faithful, definably irreducible definable G-modules.

Notation. Let (A,γ) be a definable G-module. For $i=1,\dots,m$, let (A_i,γ_i) be a definable G_i-module. We write $(G,A,\gamma)=(G_1,A_1,\gamma_1)\times\cdots\times(G_m,A_m,\gamma_m)$ if $G=G_1\times\cdots\times G_m$, $A=A_1\times\cdots\times A_m$ and for all $g=(g_1,\dots,g_m)\in G$, and all $a=(a_1,\dots,a_m)\in A$ we have $\gamma(g)(a)=(\gamma_1(g_1)(a_1),\dots,\gamma_m(g_m)(a_m))$. Recall also that $\bar{G}$ denotes $G/\mathrm{Ker}\,\gamma$ and we have a natural definable $\bar{G}$-module $(A,\bar{\gamma})$. Also, $\bar{A}=A/A^G$ and we have a natural definable G-module $(\bar{A},\gamma_{\bar{A}})$.

Theorem 6.1. *Let (U,γ) be a definable non-trivial G-module where U and G are infinite definably-connected definable groups. Let $V,K,W_1,\dots,W_s$ and $V_1,\dots,V_k$ be the definable subgroups of U given in Theorem 5.8. Then $k\geqslant 1$, and for $A\in\{V,K,W_1,\dots,W_s,V_1,\dots,V_k\}$, we have that $(A,\gamma_{|A})$ is a definable G-submodule of (U,γ) which is trivial for $A\in\{K,W_1,\dots,W_s\}$. Moreover, $(U/V,\gamma_{|U/V})$ is a trivial definable G-module.*

Proof. Let $A\in\{V,K,W_1,\dots,W_s,V_1,\dots,V_k\}$. By Remark 5.9, $(A,\gamma_{|A})$ is a definable G-submodule of (U,γ). For $A\in\{K,W_1,\dots,W_s\}$, Corollary 4.9 and Theorem 4.6 shows that $(A,\gamma_{|A})$ is a trivial definable G-module. Also $(U/V,\gamma_{|U/V})$ is a trivial definable G-module by Theorem 4.6.

Let B',B and $\Phi:G\times B'\to B'$ be as in Lemma 4.5. Suppose that $k=0$. Then by the paragraph above, V is contained in U^G. But by Lemma 3.14, U/U^G, B and B' are definably compact, contradicting Theorem 4.6. $\square$

Corollary 6.2. *Let U be a definable solvable group which is not abelian-by-finite. Then are definable subgroups B and C of U such that C/B is the additive group of a definable real closed field.*

Proof. Suppose that U is definably-connected and let $V, K, W_1, \ldots, W_s$, $V_1, \ldots, V_k$ be the subgroups of U given by Theorem 5.8. If V_i is not abelian for some i, then the result follows from Corollary 4.10. So suppose that V_i is abelian for each i. Then V is a definable U-module under conjugation. If for some i, V_i is a non-trivial definable U-submodule of V, then the result follows from Corollary 4.9. So suppose that V_i is a trivial definable U-submodule of V for all i. We can assume without loss of generality that $\mathcal{N}$ is $\aleph_0$-saturated. Then by Theorem 6.1, $V \subseteq Z(U)$ and $U/Z(U)$ is a definably-connected definable subgroup of U/V and so it is abelian and definably compact by Lemma 3.14. Since $U/Z(U)$ is infinite and abelian, there is an infinite monogenic definable subgroup Z of $U/Z(U)$. By Remark 3.12, let W be the definable normal subgroup of U such that $W/Z(U) = Z$. By Fact 4.7, W is abelian and W is a non-trivial definable U-module under conjugation, since $W^U = Z(U)$. But $W/W^U = Z$ is definably compact contradicting Theorem 4.6. □

Peterzil and Starchenko have shown ([22, Corollary 5.1]), assuming that $\mathcal{N}$ has definable Skolem functions, that if $\mathbb{U} = (U, \cdot)$ is a definable group which is not abelian-by-finite, then a real closed field is interpretable in $\mathbb{U}$. Here we get the following.

Corollary 6.3. *Let U be a definable group which is not abelian-by-finite. Then a real closed field is definable in* $(N, <, U, \cdot)$.

Proof. Suppose that U is definably-connected. Let $R(U)$ be the maximal definably-connected definable normal solvable subgroup of U. If $R(U)$ is abelian then it is a definable U-module under conjugation and if it is non-trivial we can apply Theorem 6.1. Otherwise we have $Z(U) = R(U)$ and $U/Z(U)$ is an infinite definably semi-simple definable group and the result follows from [19, Theorem 4.1]. If $R(U)$ is not abelian then the result follows from Corollary 6.2. □

Theorem 6.4. *Let (U, γ), G and $B = V_1 \times \cdots \times V_k$ be as in Theorem* 6.1. *Then $(\bar{G}, B, \bar{\gamma}) = (G_1, V_1, \gamma_1) \times \cdots \times (G_k, V_k, \gamma_k)$ where for each $i \in \{1, \ldots, k\}$, G_i is a definably-connected definable group definably isomorphic to an $\mathcal{I}_i$-definable subgroup of some $GL(m_i, I_i)$. Moreover (V_i, γ_i) is a faithful definable G_i-module and $\overline{V_i} = (I_i^{>_i 0_i}, 1_i, \cdot_i)^{l_i} \times (I_i, 0_i, +_i)^{n_i}$.*

Proof. Let $m_i = \dim V_i$. By [19, Corollary 2.22 and Fact 2..25] we have, after fixing a basis for the tangent space of each V_i, a definable homomorphism $G \to GL(m_1, I_1) \times \cdots \times GL(m_k, I_k)$ given by $g \mapsto (d_0(\gamma_{|V_1}(g)), \ldots, d_0(\gamma_{|V_k}(g))$ and with kernel $\operatorname{Ker} \gamma$. This shows that $\bar{G} = G_1 \times \cdots \times G_k$ where each G_i is definably isomorphic with an $\mathcal{I}_i$-definable subgroup of $GL(m_i, I_i)$. Since G is definably-connected, by Lemma 3.15, $\bar{G}$ is definably-connected and so each G_i is definably-connected. If we show that for $j \neq i$, $G_i \subseteq \operatorname{Ker} \bar{\gamma}_{|V_j}$, then, to prove the first part of the theorem, we can take $\gamma_i = \bar{\gamma}_{|V_i}$. Let $j \neq i$. If $G_i = 1$ then the claim holds trivially. So suppose that G_i is infinite and G_i is not contained in $\operatorname{Ker} \bar{\gamma}_{|V_j}$. Then V_j is a non-trivial definable G_i-module. Let $\Phi : G_i \times B' \to B'$ be as in Lemma 4.5. Then B' is a one-dimensional torsion-free ordered $\mathcal{I}_j$-definable group which, we can assume without loss of generality, has domain I_j.

If we apply Lemma 4.1 with $U=G_i$, $V=\mathrm{Ker}_{G_i}\Phi$ and $\mathcal{N}=\mathcal{I}_i$ and use the fact that $\mathcal{I}_i$ is an o-minimal expansion of a real closed field, then there is a continuous $\mathcal{I}_i$-definable embedding $\sigma: I_i \to G_i$ such that $\lim_{t\to-\infty}\sigma(t)=1$ and $\sigma(I_i)\subseteq G_i\setminus \mathrm{Ker}_{G_i}\Phi$. Let $x_0\in B'\setminus\{0\}$. For each $t\in I_i$, the map $\Phi(\sigma(t),-):B'\to B'$ is a definable automorphism of B'. Therefore there is a definable map $\tau: I_i\to B'$ such that for all $t\in I_i$, we have $\Phi(\sigma(t),\tau(t))=x_0$. Since $\mathcal{I}_i$ and $\mathcal{I}_j$ are globally orthogonal and are definable o-minimal expansions of real closed fields, there is $c\in B'$ such that $\lim_{t\to-\infty}\tau(t)=c$. By Lemma 4.5, there is $d\in B'$ such that Φ is continuous at $(1,c+d)$ and the definable function $\Phi(-,d):G\to B'$ is continuous. Then we have $\Phi(\sigma(t),\tau(t)+d)=\Phi(\sigma(t),\tau(t))+\Phi(\sigma(t),d)=x_0+\Phi(\sigma(t),d)$ and, taking the limit as $t\to-\infty$, we get $0=\Phi(1,c+d)=x_0+\Phi(1,d)=x_0$ which is a contradiction.

Consider G_i as an $\mathcal{I}_i$-definable group and consider the $\mathcal{I}_i$-definable group $V_i\rtimes_{\gamma_i}G_i$ whose center is $V_i^{G_i}\times(\mathrm{Ker}\,\gamma_i\cap Z(G_i))=V_i^{G_i}\times\{1\}$. By [17, Corollary 3.3] we have that $V_i\rtimes G_i/(V_i^{G_i}\times\{1\})$ is $\mathcal{I}_i$-definably isomorphic with an $\mathcal{I}_i$-definable subgroup of some $GL(l_i,I_i)$ and so by [19, Lemma 3.9] $\overline{V_i}=(I_i^{>_i 0_i},1_i,\cdot_i)^{l_i}\times(I_i,0_i,+_i)^{n_i}$. □

Theorem 6.5. *Let $\mathcal{I}=\mathcal{I}_i$, $H=G_i$ and $(A,\gamma)=(\overline{V_i},\gamma_{i|\overline{V_i}})$ be as in Theorem 6.4. Then $A=A_0\times A_1\times\cdots\times A_m$ where $(A_0,\gamma_{|A_0})$ is the maximal trivial $\mathcal{I}$-definable H-submodule of (A,γ), and for each $j\in\{1,\ldots,m\}$, we have that $(A_j,\gamma_{|A_j})$ is a definably irreducible $\mathcal{I}$-definable H-submodule of (A,γ). If $H_j=H/\mathrm{Ker}(\gamma_{|A_j})$, then $(A_j,\overline{\gamma_{|A_j}})$ is a I-semi-algebraic faithful and definably irreducible H_j-module and $H_j/Z(H_j)$ is a direct product of I-semi-algebraic I-semi-algebraically simple groups which are not abelian.*

Proof. Since by Theorem 6.4, A is a direct product of copies of the additive group and the multiplicative group of I, we have $A=A_0\times A_1\times\cdots\times A_m$ where $(A_0,\gamma_{|A_0})$ is the maximal trivial $\mathcal{I}$-definable H-submodule of (A,γ), and for each $j\in\{1,\ldots,m\}$, we have that $(A_j,\gamma_{|A_j})$ is a definably irreducible $\mathcal{I}$-definable H-submodule of (A,γ).

Let $H_j=H/\mathrm{Ker}(\gamma_{|A_j})$. Then $(A_j,\overline{\gamma_{|A_j}})$ is a faithful and definably irreducible H_j-module and by [12, Proposition 1.3], $(A_j,\overline{\gamma_{|A_j}})$ is a I-semi-algebraic faithful and definably irreducible H_j-module and $H_j/Z(H_j)$ is a direct product of I-semi-algebraic I-semi-algebraically simple groups which are not abelian. □

Corollary 6.6. *Suppose that (U,γ) is a faithful and definably irreducible definable G-module where U and G are infinite definably-connected definable groups. Then there is a definable o-minimal expansion $\mathcal{I}$ of a real closed field $(I,0,1,+,\cdot)$ such that U is definably isomorphic to $(I,0,+)^n$, the definable group G is definably isomorphic to an $\mathcal{I}$-definable subgroup of $GL(n,I)$ and (U,γ) is a I-semi-algebraic faithful and definably irreducible G-module. Moreover, $G/Z(G)$ is a direct product of I-semi-algebraic I-semi-algebraically simple groups which are not abelian.*

Proof. Since (U,γ) is a faithful and definably irreducible definable G-module, it is non-trivial, $\bar{U}=U$, $\bar{G}=G$ and by Theorem 6.1, we have $U=V=V_1$. Let $\mathcal{I}=\mathcal{I}_1$. Then by Theorem 6.4, G is definably isomorphic to an $\mathcal{I}$-definable subgroup of $GL(n,I)$ where $n=\dim U$ and U is definably isomorphic to $(I^{>0},1,\cdot)^r\times(I,0,+)^s$. We now show that U is definably isomorphic to $(I,0,+)^n$. If $(I^{>0},1,\cdot)$ is definably isomorphic to $(I,0,+)$ then we may assume that $r=0$ and $s=n$. Otherwise, $(I^{>0},1,\cdot)^r$ is a definable G-submodule of U and so either $r=0$ or $r=n$. If $r=0$, then $s=n$ and we are done. So assume that U is definably isomorphic to $(I^{>0},1,\cdot)^n$. Since $\mathscr{I}$ is an o-minimal expansion of a real closed field, $(I^{>0},1,\cdot)$ is $\mathscr{I}$-definably isomorphic to an ordered $\mathscr{I}$-definable group $(I,0,*,<)$ with respect to which $\mathscr{I}$ is linearly bounded. This is because $(I^{>0},1,\cdot)$ is not $\mathscr{I}$-definably isomorphic to $(I,0,+)$. But then, $(I,0,*)^n$ is a faithful and definably irreducible definable G-module contradicting Corollary 4.9.

The rest of the result follows from Theorem 6.5. □

6.2. The Lie–Kolchin–Mal'cev theorem

Let G be a definable group and X a subset of G. By DCC on definable subgroups, the intersection $d(X)$ of all definable subgroups of G containing X is a definable subgroup of G which we call it the *definable subgroup of G generated by X*.

Lemma 6.7. *Let G be a definable group. Then the following holds*: (1) *The operator d is a closure operator i.e., for all subsets X,Y of G we have $X\subseteq d(X)$, if $Y\subseteq X$ then $d(Y)\leqslant d(X)$ and $d(d(X))=d(X)$.* (2) *If the elements of $X\subseteq G$ commute with each other, then $d(X)$ is abelian.* (3) *If a subgroup $A\leqslant G$ normalises the subset $X\subseteq G$, then $d(A)$ normalises $d(X)$.* (4) *If $X,Y\leqslant G$ then $[d(X),d(Y)]\leqslant d([X,Y])$.*

In particular, by (4), *a subgroup $H\leqslant G$ is solvable* (*resp., nilpotent*) *of class n* iff *$d(H)$ is also solvable* (*resp., nilpotent*) of class n.

Proof. (1) is trivial. For (2) and (3) see the proof of [2, Lemma 5.35]. As for (4), the proof in [2] for the finite Morley rank analogue (see [2, Corollary 5.38 and Lemma 5.37] works in our case using the following result (which is a consequence of DCC): if G is a definable group with $H\lhd G$, $H\leqslant A\leqslant G$ and $H\subseteq Y\subseteq G$ satisfy $A/H=C_{G/H}(Y/H)$, then A is definable. □

Lemma 6.8. *Let G be a definable group.* (1) *If G is definably-connected then every finite normal subgroup is contained in $Z(G)$. If $Z(G)$ is finite then $G/Z(G)$ is centreless.* (2) *If G is infinite and nilpotent then $Z(G)$ is infinite.* (3) *If G is infinite solvable but not nilpotent then G has an infinite proper maximal normal definable subgroup H such that G/H is abelian.*

Proof. (1) is the o-minimal analogue of [15, Corollary 1] and [2, Lemma 6.1]. The proof is the same. (2) is the o-minimal analogue of [2, Lemma 6.2]; again the proof is the same. (3) is proved by an argument contained in the proof of [20, Theorem 2.12]. □

We are now ready to prove the o-minimal version of the Lie–Kolchin–Mal'cev theorem. The proof is a modification of that in [15] for the case of finite Morley rank. Before we proceed, recall that, if U is a group, then $U^{(1)}=[U,U]$ and $U^{(2)}=[U^{(1)},U^{(1)}]$.

Theorem 6.9. *If U is a definably-connected definable solvable group, then $U^{(1)}$ is a $\bigvee$-definable nilpotent normal subgroup and $d(U^{(1)})$ is a definable nilpotent normal subgroup.*

Proof. Let U be a definably-connected, definable solvable group of minimal dimension which is a counter-example to the theorem. So neither $U^{(1)}$ nor $d(U^{(1)})$ is nilpotent.

Claim 1. *We can assume that $Z(U)=Z(U^{(1)})=1$.*

Proof of Claim 1. The fact that we may assume $Z(U)=1$ follows from $(U/Z(U))^{(1)}=U^{(1)}Z(U)/Z(U)\simeq U^{(1)}/U^{(1)}\cap Z(U)\supseteq U^{(1)}/Z(U^{(1)})$. This is because $U^{(1)}\cap Z(U)\leqslant Z(U^{(1)})$. If $Z(U)$ has positive dimension, then $(U/Z(U))^{(1)}$, $U^{(1)}/Z(U^{(1)})$ and $U^{(1)}$ are nilpotent. So $Z(U)$ has dimension zero and we can substitute U by $U/Z(U)$ which is centreless by Lemma 6.8.

By Lemma 3.2 $U/C_U(U^{(1)})$ is definable. We have: $(U/C_U(U^{(1)}))^{(1)}=U^{(1)}C_U(U^{(1)})/C_U(U^{(1)})\simeq U^{(1)}/U^{(1)}\cap C_U(U^{(1)})=U^{(1)}/Z(U^{(1)})$. If $C_U(U^{(1)})$ has positive dimension, then $(U/C_U(U^{(1)}))^{(1)}$ is nilpotent and so $U^{(1)}$ is also nilpotent. Therefore, $C_U(U^{(1)})$ has dimension zero and by Lemma 6.8 we have $Z(U^{(1)})\subseteq C_U(U^{(1)})\subseteq Z(U)$. □

Claim 2. *$U^{(1)}$ and $d(U^{(1)})$ are torsion-free.*

Proof of Claim 2. Clearly, U is not definably compact, for otherwise by Corollary 4.8, it would be abelian. So by Theorem 5.8, U has a maximal definable normal subgroup W with no definably compact parts. By Remark 3.13, U/W is a definable extension of U/V by K, where K is the maximal definably-connected definably compact normal definable subgroup of U and $V=K\times W$. Since W and U are definably-connected, by Lemma 3.15, U/W is also definably-connected. Therefore, by Lemmas 3.14 and 3.15, U/W is a definably compact definably-connected definable solvable group and so by Corollary 4.8 it is abelian. Therefore we have $U^{(1)}\leqslant d(U^{(1)})\leqslant W$. We now show that W is torsion-free. In fact, by Theorem 5.8 $W=W_1\times\cdots\times W_s\times V_1\times\cdots\times V_k$. Each W_i is clearly torsion-free since it is a direct product of one-dimensional definably-connected torsion-free definable groups. So it is enough to show that each V_i is torsion-free. But V_i is an $\mathscr{I}_i$-definable solvable group with no $\mathscr{I}_i$-definably compact parts. We prove the result by induction on $\dim V_i$. If $\dim V_i=1$, then the result is clear again. If $\dim V_i>1$, then we have an $\mathscr{I}_i$-definable normal proper subgroup H of V_i such that V_i/H and H are $\mathscr{I}_i$-definable solvable groups with no $\mathscr{I}_i$-definably compact parts. Since $\dim H, \dim V_i/H<\dim V_i$, both H and V_i/H are torsion-free by the induction hypothesis. Suppose that $x\in V_i$ has finite order. Then its image in V_i/H has finite order, so it is the identity and therefore, $x\in H$ and again x is the identity. So V_i is torsion-free. □

Claim 3. *There is an infinite definable abelian normal subgroup A of U which is a definably irreducible, faithful definable $U/C_U(A)$-module under conjugation.*

Proof of Claim 3. Since U is not nilpotent, by Lemma 6.8, U has an infinite proper maximal normal definable subgroup X such that U/X is abelian. Therefore, $d(U^{(1)})$

is an infinite definable normal proper subgroup of U and so $U^{(2)} \subseteq d(U^{(1)})^{(1)} \subseteq d(d(U^{(1)})^{(1)})$ is nilpotent and infinite. Otherwise $U^{(2)}$ would be finite and since, by Claim 2, $U^{(1)}$ is torsion-free, $U^{(2)}=1$ and $U^{(1)}$ would be abelian. Now by Lemma 6.8, $Z(d(d(U^{(1)})^{(1)}))$ is infinite. Now let A be an infinite definable normal subgroup of U contained in $Z(d(d(U^{(1)})^{(1)}))$ and minimal for these properties. Note that we have $U^{(2)} \leqslant C_U(A)$ and $U/C_U(A)$ is infinite because otherwise we would have $A \leqslant Z(U) = 1$. By minimality of A, we see that A is a definably irreducible, faithful definable $U/C_U(A)$-module under conjugation. □

Corollary 6.6, $U/C_U(A)$ is abelian (since it is solvable) and therefore we have $1 = (U/C(U(A))^{(1)} = U^{(1)}C_U(A)/C_U(A) \simeq U^{(1)}/C_{U^{(1)}}(A)$. Hence, $U^{(1)} = C_{U^{(1)}}(A)$ i.e., $A \leqslant Z(U^{(1)}) = 1$ contradicting Claim 3. □

We finish this subsection with the following result on definable nilpotent groups. Recall that a group G is the central product of two subgroups H and K if $G = HK$ with H and K normal and $H \cap K \leqslant Z(G)$. We denote this by $G = H * K$. We say that a group H is *divisible* if for every $n \in \mathbb{N}$ and every $x \in H$ there is $y \in H$ such that $y^n = x$.

Theorem 6.10. *Let B be a definable nilpotent group. Then $B = B^0 * F$ for some finite subgroup F and the definably-connected component B^0 of B is divisible.*

Proof. We will first prove the result for the case $A = B^0 \subseteq Z(B)$. So suppose that this holds. It is clear that A is divisible: for every $m \in \mathbb{N}$, the kernel Ker$[m]$ of the multiplication by m homomorphism $[m] : A \to A$, is a definable subgroup of A with bounded exponent, and therefore by [27, Corollary 5.8], it is finite and so $\dim(mA) = \dim A$ and $mA = A$ because A is definably-connected.

By Corollary 3.11, there is a definable extension $1 \to A \to B \xrightarrow{j} G \to 1$ with a definable section $s : G \to B$. Let c be the corresponding definable 2-cocycle and let $\gamma : G \times A \to A$ given by $\gamma(g)(a) = < s(g) > (a)$ be the corresponding definable G-module structure on A. By Proposition 3.27, we can assume without loss of generality that B is a definable group with domain $A \times G$ and group operation given by equation (7) i.e., for all $a, b \in A$ and for all $x, y \in G$, $(a,x)(b,y) = (a + \gamma(x)(b) + c(x,y), xy)$. Let $n = |G|$. Since Ker$[n]$ is a definable normal subgroup of A and B, by Remark 3.13 we have a definable extension $1 \to nA \to nB \xrightarrow{l} G \to 1$ such that $l \circ [n] = j$ and nc is a corresponding definable 2-cocycle. For each $g \in G$, let $b(g) = \sum_{k \in G} c(g,k) \in A$. Then we have $0 = \gamma(g)(c(h,k)) - c(gh,k) + c(g,hk) - c(g,h)$ by Eq. (6). Taking the sum over elements of G, (note that $\sum_{k \in G} c(g,hk) = \sum_{k \in G} c(g,k)$) we obtain $nc(g,h) = \gamma(g)(b(h)) - b(gh) + b(g)$. Since A is divisible, there is a definable map $a : G \to A$ such that for all $g \in G$, we have $b(g) = na(g)$. Then $nc(g,h) = \gamma(g)(na(h)) - na(gh) + na(g)$. It follows from this that nc is the coboundary of na and so, by Proposition 3.28, we have $nB = nA \rtimes G$. So G is a definable subgroup of nB. Let $F_1 = [n]^{-1}(G)$. Then by Remark 3.12, F_1 is a normal finite definable subgroup of nB and we have $nB = (nA) * F_1$. Let $F = [n]^{-1}(F_1)$. Then by Remark 3.12, F_1 is a normal finite definable subgroup of B and $B = A * F$.

Let B be a counterexample to the theorem of minimal dimension.Then B is infinite and by the above, B^0 is not contained in $Z(B)$. Moreover, $Z(B)^0$ is infinite (otherwise $Z(B)$ is finite, contradicting Lemma 6.8(2)). Also $B/Z(B)^0$ is infinite (otherwise $\dim Z(B) = \dim Z(B)^0 = \dim B$ and $B^0 = Z(B)^0 \subseteq Z(B)$) with $\dim(B/Z(B)^0) < \dim B$. Therefore $B/Z(B)^0 = (B/Z(B)^0)^0 * F$. Let H and K be definable normal subgroups of B such that $H/Z(B)^0 = (B/Z(B)^0)^0$ and $K/Z(B)^0 = F$. We have $K \neq B$, $\dim K < \dim B$ and so $K = K^0 * F_1$. Now we have $B = (K^0 H) * F_1$ and by Exercise 14 on p. 6 in [2], $K^0 H$ is divisible and therefore, also definably-connected, i.e., $K^0 H = B^0$. □

7. Existence of strong definable choice

7.1. Existence of strong definable choice

Here we finally prove that definable groups have strong definable choice. By a definable topological space $A \subseteq (N \cup \{-\infty, +\infty\})^m$, we mean a definable set $A \subseteq (N \cup \{-\infty, +\infty\})^m$ with a uniformly definable topology i.e., there is a definable family $\{O(a,x) : a \in A,\ x \in X\}$ of definable subsets of A such that every $a \in A$, $\{O(a,x) : x \in X\}$ is a uniformly definable system of definable open neighbourhoods of a. For example, a definable group is a Hausdorff definable topological space.

Lemma 7.1. *Let $A \subseteq (N \cup \{-\infty, +\infty\})^m$ be a Hausdorff definable topological space. Let $\{T(x) : x \in X\}$ be a definable family of non-empty definable subsets of A such that for each $x \in X$ and for every definable map $\alpha : (c,d) \subseteq N \to T(x)$, where $-\infty \leqslant c < d \leqslant +\infty$, the limit $\lim_{t \to d^-} \alpha(t)$ exists and is an element of $T(x)$. Then there is a strong definable choice $t : X \to A$ for the definable family $\{T(x) : x \in X\}$.*

Proof. For each $i = 0, \ldots, m-1$ let $\pi^i : (N \cup \{-\infty, +\infty\})^m \to (N \cup \{-\infty, +\infty\})^{m-i}$ be the projection onto the first $m - i$ coordinates. If $a \in \pi^i(A)$ and $i = 1, \ldots, m-1$, let $F_i(a) = \{b \in N : (a,b) \in \pi^{i-1}(A)\}$, $S_i(x) = \pi^i(T(x))$ and $B_i = \{(x,a) : x \in X,\ a \in S_i(x)\}$. Note that, if $T(x) = T(y)$ then $S_i(x) = S_i(y)$.

For $x \in X$, let $S_0(x) = T(x)$, $B_0 = \{(x,y) : x \in X,\ a \in S_0(x)\}$ and let $k_0 : B_0 \to A$ be given by $k_0(x,a) = a$. The function $k_0 : B_0 \to A$ is a definable function such that for all $(x,a) \in B_0$, $k_0(x,a) \in T(x)$. Also for each $x \in X$, the map $k_0(x,-) : S_0(x) \to T(x)$ is a definable injective map and, if $T(x) = T(y)$, then $k_0(x,-) = k_0(y,-)$.

Suppose that for $i = 0, \ldots, l < m-1$ we have constructed a definable function $k_i : B_i \to A$ with the required properties. We will construct a definable function $k_{l+1} : B_{l+1} \to A$ with the same properties. Let $(x,a) \in B_{l+1}$. Then $a \in S_{l+1}(x)$ and so $\{a\} \times F_{l+1}(a) \subseteq S_l(x)$. Define $k_{l+1} : B_{l+1} \to A$ by $k_{l+1}(x,a) = \sup k_l(x, \{a\} \times F_{l+1}(a))$ where the supremum in $T(x)$ is taken with respect to the definable ordering induced by $k_l(x,-)$, which is an injective definable map from $S_l(x)$ into $T(x)$, from the natural ordering of $\{a\} \times F_{l+1}(a)$. By the hypothesis on $T(x)$, the function k_{l+1} is well defined and for all $(x,a) \in B_{l+1}$, $k_{l+1}(x,a) \in T(x)$. By the hypothesis on k_l, for every $x, y \in X$, we have $k_{l+1}(x,-) : S_{l+1}(x) \to T(x)$ is a definable injective map and, if $T(x) = T(y)$, then $k_{l+1}(x,-) = k_{l+1}(y,-)$.

Note that for every $x \in X$, $S_{m-1}(x) \subseteq N \cup \{-\infty, +\infty\}$. Define $t : X \to A$ by $t(x) = \sup\{k_{m-1}(x,a) : a \in S_{m-1}(x)\}$ where the supremum in $T(x)$ is taken with respect to the definable ordering induced by $k_{m-1}(x,-)$. This is an injective definable map from $S_{m-1}(x)$ into $T(x)$, from the natural ordering of $S_{m-1}(x)$. By the hypothesis on $T(x)$, t is well defined and for all $x \in X$, $t(x) \in T(x)$. By the hypothesis on k_{m-1}, for every $x, y \in X$, if $T(x) = T(y)$, then $t(x) = t(y)$. □

Theorem 7.2. *Let U be a definable group and let $\{T(x) : x \in X\}$ be a definable family of non-empty definable subsets of U. Then there is a definable function $t : X \to U$ such that for all $x, y \in X$ we have $t(x) \in T(x)$ and if $T(x) = T(y)$ then $t(x) = t(y)$.*

Proof. Let $R(U)$ be the maximal definable solvable normal subgroup of U. Then by Corollary 3.11, we have a definable extension $1 \to R(U) \to U \xrightarrow{l} U/R(U) \to 1$ with a definable section $s : U/R(U) \to U$. By Proposition 3.23, U is definably isomorphic with a definable group with domain $R(U) \times U/R(U)$. By Fact 2.2(iii), if we show that $R(U)$ and $U/R(U)$ have strong definable choice, it will follow that U has strong definable choice. But since $U/R(U)$ is definably semi-simple, it follows using [19, Theorem 4.1] (i.e, Theorem 12 here), Remark 2.4 and Fact 2.2(iii) that $U/R(U)$ has strong definable choice. Therefore, we may assume that U is a definable solvable group.

By Theorem 5.8 and Corollary 3.11 we have a definable extension $1 \to V \to U \xrightarrow{l} U/V \to 1$ with a definable section $s : U/V \to U$. By Proposition 3.23, U is definably isomorphic with a definable group with domain $V \times U/V$. Moreover, $V = K \times W_1 \times \cdots \times W_s \times V_1 \times \cdots \times V_k$ and K and U/V are definably compact definable abelian groups. So by Fact 2.2(iii), it is enough to show the theorem for definable groups of the form $W_1 \times \cdots \times W_s \times V_1 \times \cdots \times V_k$ and for definably compact definable abelian groups. But, by Remark 2.4 and Fact 2.2(iii), definable groups of the form $W_1 \times \cdots \times W_s \times V_1 \times \cdots \times V_k$ have strong definable choice, so we may assume that U is a definably compact definable abelian group.

Let $\{T(x) : x \in X\}$ be a definable family of non-empty definable subsets of U. Then, the family $\{\overline{T(x)} : x \in X\}$, where $\overline{T(x)}$ is the closure of $T(x)$ in U, is a definable family of non-empty definably compact definable subsets of U. By Lemma 7.1, there is a strong definable choice $l : X \to U$ for the definable family $\{\overline{T(x)} : x \in X\}$. Let O be the definable neighbourhood of 1 in U which has strong definable choice given by Lemma 2.3. And consider the definable family $S = \{S(x) : x \in X\}$ of non-empty definable subsets of O where $S(x) = \{z \in O : l(x)z \in l(x)O \cap T(x)\}$. Note that if $T(x) = T(y)$ then $S(x) = S(y)$. Let s be a strong definable choice for S. Then clearly, $t : X \to U$ given by $t(x) = l(x) \cdot s(x)$ is a strong definable choice for $\{T(x) : x \in X\}$. □

Corollary 7.3 below was also proved in [22, Theorem 1.1 and Corollary 5.2], but under the assumption that $\mathcal{N}$ has definable Skolem functions and using the theory of $\bigvee$-definable groups.

Corollary 7.3. *Let A and B be definable abelian groups. Then the following hold.*

(1) *If there is an infinite definable family of definable homomorphisms from A into B, then there is a definable real closed field whose additive group is definably*

isomorphic to a definable subgroup of B and a quotient of definable subgroups of A.

(2) *If A is infinite, defined over* $a \in N^{\urcorner}$ *and there is a definable subgroup of A which is not defined over* $\operatorname{acl}(a)$ (*that is, there is an infinite definable family of definable subgroups of A*), *then there is a definable real closed field whose additive group is definably isomorphic to a quotient of definable subgroups of A.*

Proof. (1) Let $\gamma : S \times A \to B$ be an infinite definable family of definable homomorphisms from A into B. Then by [22, Lemma 2.17], there is $\{a_1, \ldots, a_n\} \subseteq A$ such that for $s \in S$, $\gamma(s)$ is determined by its values on this finite set. Therefore, we can identify S with a definable subset of $A \times \cdots \times A$ (n times) and so, by Theorem 7.2 and Fact 2.2(i), S has strong definable choice. Now the rest of the proof is obtained by adapting the proof of (1) in [22].

(2) The argument in the proof of [22, Corollary 5.2] together with Theorem 7.2 reduces the proof of (2) to case (1). □

7.2. More on definable extensions

In this subsection we apply Theorem 7.2 to the theory of definable group extensions.

Definition 7.4. Let (A, θ) be a definable G-kernel. We say that $\alpha, \beta \in \theta$ are *definably related* if there is a definable function $k : G \to A$ such that for all $x \in G$ $\beta(x) = \langle k(x) \rangle \alpha(x)$. In this case we have $h_\beta(x, y) = k(x)\alpha(x)(k(y))h_\alpha(x, y)k(xy)^{-1}$ for all $x, y \in G$.

Remark 7.5. By Theorem 7.2, any two $\alpha, \beta \in \theta$ are definably related. In fact, since $\imath(\beta(x)) = \imath(\alpha(x))$ for all $x \in G$, for each $x \in G$ we have a non-empty definable subset $T(x) = \{b \in A : \forall a \in A,\ \beta(x)(a) = (< b > \alpha(x))(a)\}$ of A and $\{T(x) : x \in G\}$ is a definable family; by Theorem 7.2, there is a strong definable choice $k : G \to A$ and we have $\beta(x) = \langle k(x) \rangle \alpha(x)$ for all $x \in G$.

Using Remark 7.5, the proof of the next result is just like the proof of [6, Theorem 10.1].

Remark 7.6. There is a canonical map from $K_{\mathcal{N}}(G, B)$ into $H^3_{\mathcal{N}}(G, B, \theta_0)$, sending (A, θ) into $c_{(A,\theta)}$ and $(A, \theta) \in EK_{\mathcal{N}}(G, B)$ if and only if $c_{(A,\theta)} = 1$. By Remark 3.30, this map is a homomorphism with kernel $EK_{\mathcal{N}}(G, B)$.

8. Definable rings

In this section we apply our results on definable abelian groups to describe definable rings. We start by recalling some facts about definable rings.

Let U be a definable ring. Then by [17, Lemma 4.1], U can be equipped with a unique definable manifold structure making the ring into a topological ring and, by [19, Lemma 1.11], definable homomorphisms between definable rings are topological homomorphisms. In fact, by [17, Lemma 4.1], if $\mathcal{N}$ is an o-minimal expansion of a real closed field then U equipped with the above unique definable manifold structure is a C^p ring for all $p \in \mathbb{N}$ and by [19, Lemma 1.11], definable homomorphisms between definable rings are C^p homomorphisms for all $p \in \mathbb{N}$.

It follows from the DCC for definable groups, that U satisfies the descending chain condition (DCC) on definable left (resp., right and bi-) ideals. Let U^0 be the definable-connected component of zero in the additive group of U. Then U^0 is the smallest definable ideal of U of finite index. We say that U is definably-connected if $U^0 = U$. Finally we mention the following result, see [23, Theorem 4.1], which we generalise below.

Theorem 8.1 (Peterzil and Steinhorn [23]). *If U is an infinite definable associative ring without zero divisors, then U is a division ring and there is a one-dimensional definable subring I of U which is a real closed field such that U is either I, $I(\sqrt{-1})$, or the ring of quaternions over I.*

Theorem 8.2. *Let U be a definably-connected definable ring (not necessarily associative). Let $V, K, W_1, \ldots, W_s, V_1, \ldots, V_k$ be the additive subgroups of U given by Theorem* 5.8. *Then $A \in \{V, K, W_1, \ldots, W_s, V_1, \ldots, V_k\}$ is a definable ideal of U, $A \in \{K, W_1, \ldots, W_s\}$ and U/V are definable rings with zero multiplication and each V_i (with $i = 1, \ldots, k$) is an $\mathscr{I}_i$-definable ring whose additive group has no $\mathscr{I}_i$-definably compact parts where $\mathscr{I}_i$ is the definable o-minimal expansion of a real closed field given by Theorem* 5.8.

Proof. By Remark 5.9, $A \in \{V, K, W_1, \ldots, W_s, V_1, \ldots, V_k\}$ is a definable ideal of U.

By Corollary 7.3 (1), if $A \in \{K, W_1, \ldots, W_s, U/V\}$, then A is a definable ring with zero multiplication since, multiplication on A is continuous, induces a definable family of definable endomorphisms of A and A is definably-connected (by Lemma 3.15 in the case U/V). Finally, by construction of $\mathscr{I}_i$, V_i is a $\mathscr{I}_i$-definable ring. □

Theorem 8.3. *Let $\mathscr{I}$ be a definable o-minimal expansion of a real closed field $I = (I, 0, 1, +, \cdot)$ and let U be an $\mathscr{I}$-definable ring (not necessarily associative) whose additive group has no $\mathscr{I}$-definably compact parts. Let W and V be the additive definable subgroups of U given by Theorem* 5.10 *and such that $U = W \times V$. Then W and V are $\mathscr{I}$-definable ideals of U, the ideal W has zero multiplication. The ideal V is an $\mathscr{I}$-definable ring such that $\bar{V} = V/\mathrm{ann}_V V$ is a finitely generated I-algebra (and therefore I-definable). If $\bar{V}$ is associative, then it is $\mathscr{I}$-definably isomorphic to a finitely generated I-subalgebra of some $M_n(I)$ and has a nilpotent finitely generated ideal Z such that $\bar{V}/Z$ is $\mathscr{I}$-definably isomorphic to $\bigoplus_{j=1}^m M_{k_j}(D_j)$ where for each $j = 1, \ldots m$, D_j is either I, $I(\sqrt{-1})$, or the ring of quaternions over I.*

Proof. Since W is the maximal $\mathscr{I}$-definable additive subgroup of U which is a product of copies of the one-dimensional, torsion-free, linearly bounded $\mathscr{I}$-definable group, and since V is the maximal $\mathscr{I}$-definable additive subgroup of U which has no $\mathscr{I}$-definable additive subgroup $\mathscr{I}$-definably isomorphic to a product of copies of the one-dimensional, torsion-free, linearly bounded $\mathscr{I}$-definable group, it follows that the additive $\mathscr{I}$-definable subgroups W and V of U are invariant under every $\mathcal{I}$-definable endomorphism of U. Therefore, the additive $\mathcal{I}$-definable subgroups W and V of U are $\mathcal{I}$-definable ideals of U.

By Corollary 7.3(1), W has zero multiplication since multiplication on W is a continuous map inducing an $\mathcal{I}$-definable family of $\mathcal{I}$-definable endomorphisms of W and W is $\mathcal{I}$-definably-connected. The fact that $\bar{V}$ is $\mathcal{I}$-definably isomorphic with a finitely generate I-algebra which, if it is associative, is $\mathcal{I}$-definably isomorphic to a finitely generated I-subalgebra of some $M_n(I)$ follows from (the proof of) [17, Lemma 4.3]. By [1] there is a nilpotent finitely generated ideal Z of $\bar{V}$ such that $\bar{V}/Z$ is $\mathcal{I}$-definably semi-simple and therefore $\bar{V}/Z$ is $\mathcal{I}$-definably isomorphic to $\bigoplus_{j=1}^{m} M_{k_j}(D_j)$ where for each $j=1,\dots m$, D_j is either I, $I(\sqrt{-1})$, or the ring of quaternions over I. For details see [1, Chapter 5, Section 13, Corollary 20, Theorem 23 and Theorem 16]]. □

Definition 8.4. Recall that a *Lie ring* is an additive group L with a bilinear product (called bracket) $[x,y]$ such that for all $x,y,z \in L$ (i) $[x,x]=0$ and (ii) $[[x,y],z]+[[y,z],x]+[[z,x],y]=0$ (Jacobi identity). L is *abelian* if for all $x,y \in L$, $[x,y]=0$.

Since a definable Lie ring is a definable ring, Theorem 8.2 applies to definable Lie rings. Theorem 8.5 below is the Lie ring analogue of Theorem 8.3 and is proved in the same way using the Lie ring analogue of Lemma 4.3 in [17] i.e., let $\mathcal{I}$ be a definable o-minimal expansion of a real closed field $I=(I,0,1,+,\cdot)$ and let U be an $\mathcal{I}$-definably-connected $\mathcal{I}$-definable Lie ring of dimension n. Then the $\mathcal{I}$-definable map $D: U \to M_n(I)$ given by $D(u)=d_0(\lambda_u)$ where for $u \in U$ and $x \in U$, $\lambda_u(x)=[u,x]$, is an $\mathcal{I}$-definable Lie ring homomorphism with kernel $\mathrm{ann}_U U=\{u \in U : [u,x]=0 \text{ for all } x \in U\}$.

Theorem 8.5. *Let $\mathcal{I}$ be a definable o-minimal expansion of a real closed field $I=(I,0,1,+,\cdot)$ and let U be an $\mathcal{I}$-definable Lie ring whose additive group has no $\mathcal{I}$-definably compact parts. Let W and V be the additive definable subgroups of U given by Theorem* 5.10 *and such that $U=W\times V$. Then W and V are $\mathcal{I}$-definable Lie ideals of U, the ideal W is an abelian $\mathcal{I}$-definable Lie ring, V is an $\mathcal{I}$-definable Lie ring such that $\bar{V}=V/\mathrm{ann}_V V$ is $\mathcal{I}$-definably isomorphic to a finitely generated Lie subalgebra of some $M_n(I)$.*

9. For further reading

The following articles were not specifically cited in the text but may be of interest to the reader: Hoschchild [7,8], Loveys and Peterzil [11] and Nesin et al. [16].

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